PIECEWISE SPARSE RECOVERY VIA PIECEWISE INVERSE SCALE SPACE ALGORITHM WITH DELETION RULE

Yijun Zhong and Chongjun Li
School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China
Email: zhongyijun@mail.dlut.edu.cn, chongjun@dlut.edu.cn

Abstract

In some applications, there are signals with piecewise structure to be recovered. In this paper, we propose a piecewise ISS (P_ISS) method which aims to preserve the piecewise sparse structure (or the small-scaled entries) of piecewise signals. In order to avoid selecting redundant false small-scaled elements, we also implement the piecewise ISS algorithm in parallel and distributed manners equipped with a deletion rule. Numerical experiments indicate that compared with aISS, the P_ISS algorithm is more effective and robust for piecewise sparse recovery.

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Key words: Inverse scale space, Piecewise sparse, Sparse recovery, Small-scaled entries.

1. Introduction

In this paper, we consider recovering a sparse signal \( x^* \in \mathbb{R}^n \) from its noisy linear measurements

\[
\mathbf{b} = A\mathbf{x}^* + \mathbf{e},
\]

where \( \mathbf{b} \in \mathbb{R}^m \) is a measurement vector, \( A \in \mathbb{R}^{m \times n} \) is a measurement matrix, and \( \mathbf{e} \in \mathcal{N}(0, \sigma^2 \mathbf{I}_n) \) is Gaussian noise. The sparse vector \( x^* \) has \( s \leq m < n \) nonzero entries. A widely used method to perform this reconstruction is the Basis Pursuit, i.e., to solve the following minimization problem

\[
\min_\mathbf{x} \| \mathbf{x} \|_1, \quad \text{s.t.} \ A\mathbf{x} = \mathbf{b}.
\]

The key of recovering a signal in this setting is to find the support of the signal, i.e., find the set \( S \) satisfying \( \text{supp}(x^*) = S \), it is named as “exact support recovery”. In some applications, the signal is indeed “piecewise sparse”. For example, the problem of the decomposition of texture part and cartoon part of image in [20], i.e., \( \mathbf{b} = A_n\mathbf{x}_n + A_t\mathbf{x}_t \) where \( n \) and \( t \) represent the cartoon and texture. It is assumed that both parts can be represented in some given dictionaries, thus \( \mathbf{x}_n \) and \( \mathbf{x}_t \) are two sparse vectors. The coefficient vector \( \mathbf{x} = (\mathbf{x}_n^T, \mathbf{x}_t^T)^T \) is “piecewise” sparse vector. Another example is the problem of reconstructing a surface from scattered data in approximation space \( \mathcal{H} = \bigcup_{j=1}^N H_j \), where \( H_j \subseteq H_{j+1} \) are principal shift invariant (PSI) spaces generated by a single compactly supported function [18], the fitting surface is \( g = \sum_{i=1}^N g_i, \ g_i \in H_i \) with \( g_i = \sum_{j=1}^{n_i} c_j^i \phi_j^i \). The coefficients \( \mathbf{c} = (\mathbf{c}^1, \ldots, \mathbf{c}^N)^T \) (by \( N \)

1) the corresponding author.

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pieces \( c^i = (c_1^i, \ldots, c_n^i)^T \) is the vector to be determined. Due to the property of PSI spaces, the coefficients to be determined by \( l_1 \) minimization in [18] are “piecewise” sparse structured, i.e., each \( c^i \in \mathbb{R}^n \) is a sparse vector in \( H_i \).

To be general, we recover a sparse signal \( x = (x_1^T, \ldots, x_N^T)^T \) which is piecewise sparse structured by a partition of support set \( S = (S_i)_{i=1}^N \). Denote the corresponding partition of \( D = \{1, \ldots, n\} \) as \( D = (D_i)_{i=1}^N \). It is clear that \( S_i \subseteq D_i \). Then we recover \( N \) sub-signals \( x^i (i = 1, \ldots, N) \) simultaneously. We call this type of signal as “piecewise sparse” vector, denoted by \( (s_1, \ldots, s_N) \)-sparse vector. According to the piecewise structure of the signal \( x \), the measurement matrix \( A \) is also structured as \( A = [A_1, \ldots, A_N] \) where \( A_i \in \mathbb{R}^{m \times n_i} \). Then the linear measurements (1.1) can be rewritten as

\[
b = \sum_{i=1}^{N} A_i x_i^* + e.
\]

Based on this, we provide the definition of piecewise sparse vector:

**Definition 1.1.** Suppose the \( m \)-sample vector \( b \) is the linear superposition of \( N \) components with some additive noise,

\[
b = \sum_{i=1}^{N} b_i + e. \tag{1.3}
\]

Furthermore, assume that each \( b_i \) can be sparsely represented in a basis \( A_i \), i.e.,

\[b_i = A_i x_i, \quad i = 1, \ldots, N,
\]

where \( x_i \) is a sparse vector. We define the vector \( x = (x_1^T, \ldots, x_N^T)^T \) as a piecewise sparse vector. In particular, if the piecewise sparsity is provided, i.e., number of nonzero entries of \( x_i \) is \( s_i \) for each \( i \), then we denote the piecewise sparse vector \( x = (x_1^T, \ldots, x_N^T)^T \) as \( (s_1, \ldots, s_N) \)-piecewise sparse vector.

![Fig. 1.1. Example of block sparse vector.](image_url)

**Remark 1.1.** It is necessary to claim that the piecewise sparse vector is quite different from the block sparse vector mentioned in [14–16, 26]. A block \( s \)-sparse vector \( x = (x^T[1], \ldots, x^T[N])^T \) is assumed to have at most \( s \) blocks with nonzero entries while each block \( x[l] \) \((l = 1, \ldots, N)\) is not necessary sparse. Furthermore, a block sparse vector is not necessary sparse. See the example in [16] (Fig. 1.1). In this example, 2 nonzero blocks out of 100 blocks correspond to 200 nonzero elements out of 298 elements. A piecewise sparse vector \( x = (x_1^T, \ldots, x_N^T)^T \) is partitioned into \( N \) components and it is assumed that every \( x_i \in \mathbb{R}^{n_i} \) containing nonzero entries is sparse. See the following example in Fig. 1.2, there are 100 parts are each part has one nonzero element. It is clear that a piecewise sparse vector must be a sparse vector in general meaning.

**Remark 1.2.** Note that the sub-vectors \( x_i^* \) \((i = 1, \ldots, N)\) in equation \( b^* = \sum_{i=1}^{N} A_i x_i^* + e \) are correlated to each other, thus these sub-vectors \( x_i^* \) \((i = 1, \ldots, N)\) cannot be recovered independently.