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A C^0-WEAK GALERKIN FINITE ELEMENT METHOD FOR THE TWO-DIMENSIONAL NAVIER-STOKES EQUATIONS IN STREAM-FUNCTION FORMULATION*

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Abstract

We propose and analyze a C^0-weak Galerkin (WG) finite element method for the numerical solution of the Navier-Stokes equations governing 2D stationary incompressible flows. Using a stream-function formulation, the system of Navier-Stokes equations is reduced to a single fourth-order nonlinear partial differential equation and the incompressibility constraint is automatically satisfied. The proposed method uses continuous piecewise-polynomial approximations of degree $k+2$ for the stream-function ψ and discontinuous piecewise-polynomial approximations of degree $k+1$ for the trace of $\nabla\psi$ on the interelement boundaries. The existence of a discrete solution is proved by means of a topological degree argument, while the uniqueness is obtained under a data smallness condition. An optimal error estimate is obtained in L^2-norm, H^1-norm and broken H^2-norm. Numerical tests are presented to demonstrate the theoretical results.

Mathematics subject classification: 65N12, 65N30.
Key words: Weak Galerkin method, Navier-Stokes equations, Stream-function formulation.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded, convex, simply connected polygonal domain with boundary $\partial\Omega$. It is well-known that in two dimensions, the Navier-Stokes equations can be written in the following stream-function formulation [3, 12, 18]:

$$\begin{cases} \mu\triangle^2\psi - \dfrac{\partial}{\partial x_1}\left(\dfrac{\partial\psi}{\partial x_2}\triangle\psi\right) + \dfrac{\partial}{\partial x_2}\left(\dfrac{\partial\psi}{\partial x_1}\triangle\psi\right) = f, & \text{in } \Omega, \\ \psi = g_1, & \text{on } \partial\Omega, \\ \frac{\partial\psi}{\partial\boldsymbol{\nu}} = g_2, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where ψ is a stream-function, $\mu = Re^{-1} > 0$ is the fluid viscosity coefficient with Re denoting the Reynolds number, $f \in L^2(\Omega)$, $\boldsymbol{\nu}$ is the outward unit normal vector to $\partial\Omega$, g_1 and g_2 are

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appropriate boundary data. The stream-function ψ can be associated with the divergence-free velocity field U via

$$U=(u,v)=\mathbf{curl}\psi:=\Big(\frac{\partial\psi}{\partial x_2},-\frac{\partial\psi}{\partial x_1}\Big). \tag{1.2}$$

The attractions of the stream-function formulation are that the incompressibility constraint is automatically satisfied and there is only one unknown field to solve. A possible approach for the problem (1.1) is to use conforming finite element methods. The advantage of this approach is that convergence is easy to be guaranteed, while the disadvantage is that it requires C^1 finite elements which are quite complicated [14]. In practice, this is far from desirable. To overcome this difficulty, nonstandard methods have been developed for the problem (1.1) and also for the fourth-order elliptic problems, such as the nonconforming method [3], the C^0 interior penalty methods [2], the discontinuous Galerkin methods [18] and the mixed methods [12].

Another way to avoid constructing H^2-conforming finite elements is to use the weak Galerkin (WG) methods. The main feature of the WG method is that differential operators are approximated by weak forms as distributions and a stabilization term is added to enforce a weak continuity of the approximation functions. Thus there is no need to construct C^1 finite elements to solve (1.1). The WG method was first proposed and analyzed in [23] for general second order elliptic problems and applied to many other problems: Stokes problems [5,24], quasi-Newtonian Stokes problem [28], Navier-Stokes problem [27], Brinkman equations [25], convection-diffusion-reaction problems [6], elasticity problems [7] and wave equation [16]. Recently, the WG methods have been successfully applied to fourth order elliptic problems [4,19,20,22,26]. Roughly speaking, there are two types of WG methods for fourth order equations: C^0-WG and non-C^0-WG. The C^0-WG methods use continuous piecewise-polynomials (so it is called "C^0"), while non-C^0 methods use totally discontinuous polynomials on meshes. An advantage of the non-C^0 methods is that it allows to use polygonal or polyhedral meshes. In contrast, though C^0-WG method does not generally allow to use polygonal or polyhedral meshes, it reduces less number of unknowns due to the continuity requirement. Considering the successful application of WG to fourth order problem, it is quite natural to ask whether it is possible to extend these methods to nonlinear fourth order equations. In addition, to our best of knowledge, the researches on WG methods for Navier-Stokes equations in stream-function formulation have not yet been reported before, and this also motivates us to employ WG methods to the problem (1.1).

Our C^0-WG method uses continuous piecewise-polynomial approximations of degree $k+2$ $(k\geq 0)$ for the stream-function ψ and discontinuous piecewise-polynomial approximations of degree $k+1$ for the trace of $\nabla\psi$ on the interelement boundaries. The existence of a discrete solution is proved by means of a topological degree argument, and the solution is also unique provided a data smallness condition on f is verified. We also prove the optimal error estimate in L^2-norm, H^1-norm and broken H^2-norm, and numerical tests are provided to illustrate and confirm our theoretical analysis.

The paper is arranged as follows. In Section 2, we provide necessary notations and weak formulation for the 2D Navier-Stokes equations in stream-function formulation. In Section 3, some basic and important results are presented. In Section 4, we prove the existence and uniqueness of the approximation, and give the error estimates in broken H^2-norm. In Section 5, we establish some error estimates in H^1 and L^2-norm. Finally, we do some numerical tests in Section 6.