Journal of Computational Mathematics
Vol.38, No.2, 2020, 239–253.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1805-m2017-0081

A HIGH-ORDER ACCURACY METHOD FOR SOLVING THE FRACTIONAL DIFFUSION EQUATIONS*

Maohua Ran
School of Mathematics and Software Science, Sichuan Normal University, Chengdu 610066, China;
School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China
Email: maohuaran@163.com
Chengjian Zhang[1)]
School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China
Email: cjzhang@mail.hust.edu.cn

Abstract

In this paper, an efficient numerical method for solving the general fractional diffusion equations with Riesz fractional derivative is proposed by combining the fractional compact difference operator and the boundary value methods. In order to efficiently solve the generated linear large-scale system, the generalized minimal residual (GMRES) algorithm is applied. For accelerating the convergence rate of the iterative, the Strang-type, Chan-type and P-type preconditioners are introduced. The suggested method can reach higher order accuracy both in space and in time than the existing methods. When the used boundary value method is A_{k_1,k_2}-stable, it is proven that Strang-type preconditioner is invertible and the spectra of preconditioned matrix is clustered around 1. It implies that the iterative solution is convergent rapidly. Numerical experiments with the absorbing boundary condition and the generalized Dirichlet type further verify the efficiency.

Mathematics subject classification: 65F08, 65M06, 65M12.
Key words: Boundary value method, Circulant preconditioner, High accuracy, Generalized Dirichlet type boundary condition.

1. Introduction

Consider the following problem of fractional-in-space diffusion equations

$$\frac{\partial u(x,t)}{\partial t}=\kappa\frac{\partial^{\gamma}u(x,t)}{\partial|x|^{\gamma}}+f(x,t),\qquad (x,t)\in(a,b)\times(t_0,T],\tag{1.1}$$

with the initial value condition

$$u(x,t_0)=u_0(x),\ x\in(a,b),\tag{1.2}$$

and the generalized Dirichlet type boundary condition

$$u(x,t)=\psi(x,t),\qquad x\notin(a,b),t\in(t_0,T],\tag{1.3}$$

* Received March 27, 2017 / Revised version received September 19, 2017 / Accepted May 7, 2018 / Published online May 21, 2019 /
1) Corresponding author

where $\kappa > 0$ is the diffusion coefficient, $1 < \gamma \le 2$, $u_0(x)$ is given smooth function, the Riesz fractional derivative of order γ is defined by

$$\frac{\partial^\gamma u(x,t)}{\partial |x|^\gamma} = -\frac{1}{2\cos(\pi\gamma/2)} \left[{}_aD_x^\gamma u(x,t) + {}_x D_b^\gamma u(x,t) \right], \tag{1.4}$$

where ${}_aD_x^\gamma u(x,t)$ and ${}_xD_b^\gamma u(x,t)$ are the left and right Riemann-Liouville fractional derivatives, respectively. Function $\psi(x,t)$ should satisfy that there exist positive constants M and C such that

$$\frac{|\psi(x,t)|}{|x|^{\gamma-\epsilon}} < C \quad \text{for positive small } \epsilon \text{ when } |x| > M, \tag{1.5}$$

see [1]. One of the most popular cases is $\psi(x,t) \equiv 0$, which is the so called absorbing boundary condition, and implying that the particle is killed whenever it leaves the interval (a,b). The above problems arise frequently from anomalous diffusion, turbulence, biology and the other scientific fields, see, for example [2–6] and references therein.

In the recent years, around this particular case of $\psi(x,t) \equiv 0$, with the arguments such as shifted Grünwald formula [7], fractional central difference [8, 9], Lubich's operator [10] and their weighted averages (see, e.g., [11]), some numerical methods for solving problem (1.1)-(1.3) have been presented. For example, Haghighi et al. [3] proposed the explicit and implicit Euler methods, Çelik and Duman [2] suggested a Crank-Nicolson method, Lin et al. [12] derived the preconditioned conjugate gradient normal residual method and preconditioned generalized minimal residual method, Ding et al. [4] developed a fourth-order approximation for Riesz fractional derivative and then applied it to the problem (1.1)-(1.3). These methods can arrive at the high-order accuracy in space. Nevertheless, owing to only Euler method and Crank-Nicolson method were used to the time discretization, their accuracies in time need to be improved. In order to raising up the temporal accuracy of numerical methods, Gu and Lei [13, 14] considered a class of BVMs with Strange-type preconditioner for the two-side fractional-in-space diffusion equation, respectively. However, their method has only order one in space.

In view of the above research, for the more general problems (1.1)-(1.3), it is interesting to construct the numerical methods with higher-order accuracies both in time and space. Hence, in the present paper, we will focus on this topic. For the space-discretization, we will consider the fractional compact difference operator with high-order accuracy [15, 16]. It is remarkable that the compact difference method has successively been applied to solve two-dimensional space fractional Schrödinger equation. For the time-discretization, we will take use of the boundary value methods (BVMs) based on linear multistep formulaes [17]. It has been applied recently to the distributed order sub-diffusion equation [18]. Comparing the underlying linear multistep formulaes, this type of BVMs can arrive at higher-order accuracy and possess A_{k_1,k_2}-stability when applied to ordinary differential equations (ODEs). Although they need a large computational cost, this may be improved effectively by introducing some circulant preconditioners, such as the Strang-type, Chan-type, P-type preconditioners and so forth, see, for example [19–23]. Moreover, by applying the GMRES algorithm to the generated preconditional systems, the computational efficiency of the BVMs can be further raised.

This paper is organized as follows. In Section 2, we briefly review the underlying BVMs applied to the linear ODEs, which is a foundation that we construct the compact BVMs for the fractional-in-space diffusion equations. In Section 3, a fully discrete difference scheme for the problem (1.1)-(1.3) is derived by combining the underlying BVMs and the fractional compact difference method. In Section 4, in order to raise the computational efficiency of the