Explicit $H^1$-Estimate for the Solution of the Lamé System with Mixed Boundary Conditions

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Abstract. In this paper we consider the Lamé system on a polygonal convex domain with mixed boundary conditions of Dirichlet-Neumann type. An explicit $L^2$ norm estimate for the gradient of the solution of this problem is established. This leads to an explicit bound of the $H^1$ norm of this solution. Note that the obtained upper-bound is not optimal.

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1 Introduction

Let $\Omega$ be a bounded open connected subset of $\mathbb{R}^2$. The static equilibrium of a deformable structure occupying $\Omega$ is governed by the Lamé linear elasto-static system, see [1]. In this paper, we restrict the study to a convex domain $\Omega$ whose boundary has a polygonal shape that possesses $m+1$ edges with $m \geq 2$. We denote $\Gamma = \bigcup_{i=0}^{m} \Gamma_i$ its boundary and $d(\Omega)$ its diameter. Moreover, we assume that all the edges $\Gamma_i$ have strictly positive measure. The system under consideration is given by

$$
\begin{aligned}
Lu &= f \quad \text{a.e in } \Omega, \\
\sigma(u) \cdot \vec{n}_i &= g_i \quad \text{on } (\Gamma - \Gamma_0) \cap \Gamma_i, \ 1 \leq i \leq m, \\
u &= 0 \quad \text{on } \Gamma_0.
\end{aligned}
$$

(1.1)

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We need to assume that the edges $\Gamma_i$ which form the boundary $\Gamma$ fulfill a condition similar to assumption $(H_2)$ in ([2], Theorem 2.3). Actually, for our purpose, a stronger condition is needed and it is formulated in (1.5) below. The vector function $u = (u^1, u^2)$ satisfying the system (1.8) describes a displacement in the plane. In this model we impose a homogeneous Dirichlet condition on $\Gamma_0$ and a Neumann condition on the remaining part of the boundary. The equality on the boundary is understood in the sense of the trace. We denote $L$ the Lamé operator defined by

$$Lu := -\text{div}\sigma(u) = -\text{div}[2\mu\varepsilon(u) + \lambda\text{Tr}\varepsilon(u)\text{Id}].$$  \hfill (1.2)

We assume the data functions $f$ and $g$ at the right hand sides to satisfy $f \in [L^2(\Omega)]^2$ and $g \in [H^\frac{1}{2}(\Gamma_0)]^2$. The vector $\overrightarrow{n}_i$ represents the outside normal to $\Gamma_i$. We write $\mu$ and $\lambda$ the Lamé’s coefficients. We place ourselves in the isotropic framework, the deformation tensor $\varepsilon$ is defined by

$$\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla^t u).$$  \hfill (1.3)

The weak form of problem (1.1) is (see [1,3]): Find $u \in V$ such that $\forall v \in V$

$$\int_\Omega 2\mu\varepsilon(u)\varepsilon(v) + \lambda\text{div}u \text{ div}v \text{d}x = \int_\Omega fv \text{d}x + \int_{\Gamma_{-\Gamma_0}}gv \text{d}\sigma(x),$$  \hfill (1.4)

where

$$V = \{ v \in [H^1(\Omega)]^2; \ v = 0 \text{ on } \Gamma_0 \}.$$  

The existence and uniqueness issue of the solution of (1.4) in $V$ is classic, (see [3]).

If we denote $\theta$ the interior angle between the edges $\Gamma_j$ and $\Gamma_k$, $0 \leq j, k \leq m$ such that $\overline{\Gamma_j} \cap \overline{\Gamma_k} \neq \emptyset$ and if we denote $\gamma$ the interior angle between the Neumann part of the boundary $\Gamma_N := \Gamma - \Gamma_0$ and the Dirichlet part of the boundary $\Gamma_D := \Gamma_0$, then we impose

$$0 < \theta < \pi, \quad 0 < \gamma < \pi.$$  \hfill (1.5)

The reason behind this assumption on the boundary is to get a better regularity of the solution of the weak problem (1.4). Precisely, in that case we have, following ([2], Theorem 2.3) stated at the bottom of page 330, $u \in [H^{\frac{3}{2}+\epsilon}(\Omega)]^2$ for some positive $\epsilon > 0$, which implies in particular, using the appropriate Sobolev embedding and since $\Omega$ is a locally Lipschitz domain, see part II of ([4], Theorem 4.12, page 85), that $u \in [C^{0,\frac{1}{2}+\epsilon}(\overline{\Omega})]^2$ i.e. $u$ is $(\frac{1}{2} + i)$–holder continuous. One should notice that condition (1.5) are met since the domain considered in our case is convex. Let us denote

$$||\varepsilon(u)||_{0,\Omega} := \left(\int_\Omega \varepsilon(u)\varepsilon(u)\text{d}x\right)^{\frac{1}{2}}; \quad ||\nabla u||_{0,\Omega} := \left(\int_\Omega |\nabla u|^2 + |\nabla^t u|^2\text{d}x\right)^{\frac{1}{2}}.$$  

By using the second Korn inequality, see [5], the trace and the Poincaré’s inequalities, one easily gets from (1.4) the following estimate

$$||\nabla u||_{0,\Omega} \leq \frac{1}{c_k} \frac{1}{2\mu} (c_p ||f||_{0,\Omega} + c_p \lambda ||g||_{\frac{1}{2}, \Gamma_0}),$$  \hfill (1.6)