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Gradient Estimates for a Nonlinear Heat Equation Under Finsler-geometric Flow

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Abstract. This paper considers a compact Finsler manifold $(M^n,F(t),m)$ evolving under a Finsler-geometric flow and establishes global gradient estimates for positive solutions of the following nonlinear heat equation

$$\partial_t u(x,t) = \Delta_m u(x,t), \qquad (x,t) \in M \times [0,T],$$

where Δ_m is the Finsler-Laplacian. By integrating the gradient estimates, we derive the corresponding Harnack inequalities. Our results generalize and correct the work of S. Lakzian, who established similar results for the Finsler-Ricci flow. Our results are also natural extension of similar results on Riemannian-geometric flow, previously studied by J. Sun. Finally, we give an application to the Finsler-Yamabe flow.

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1 Introduction

The paper studies nonlinear heat equation

$$\partial_t u(x,t) = \Delta_m u(x,t) \tag{1.1}$$

on a compact Finsler manifold $(M^n,F(t),m)$ evolving by the Finsler-geometric flow

$$\frac{\partial}{\partial t} g(t) = 2h(t), \tag{1.2}$$

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where $(x,t) \in M \times [0,T]$, $g(t)$ is the symmetric metric tensor associated with F, and $h(t)$ is a symmetric $(0,2)$-tensor field on $(M^n, F(t), m)$. An important example would be the case where $h(t) = -Ric_{ij}(t)$ and $g(t)$ is a solution of the Finsler-Ricci flow introduced by Bao [1]. Unlike the usual Laplacian, the Finsler-Laplacian Δ_m is a nonlinear operator. For the existence, uniqueness and Sobolev regularity of a positive global solution of the nonlinear heat equation (1.1) (in the sense of distributions), we can see [2]. We will give some gradient estimates and Harnack inequalities for positive global solutions of equation (1.1).

The study of gradient estimates for the heat equation originated with the work of P. Li and S.-T. Yau [3]. They proved a space-time gradient estimate for positive solutions of the heat equation on a complete manifold. By integrating the gradient estimate along a space-time path, a Harnack inequality was derived. Therefore, Li-Yau inequality is often called differential Harnack inequality. Li-Yau type gradient estimates have been obtained for other nonlinear equations on manifolds, see for example [4-13] and the references therein. Over the past two decades, many authors used similar techniques to prove gradient estimates and Harnack inequalities for geometric flows. For instance, in [14], weakening Guenther's curvature constrains in [15] on the boundedness of the gradient of scalar curvature in addition to the boundedness of the Ricci curvature, Liu established first order gradient estimates for positive solutions of the heat equations on complete noncompact or closed Riemannian manifolds under Ricci flows. As applications, he derived Harnack type inequalities and second order gradient estimates for positive solutions. Generalizing Liu's work to general geometric flow, Sun [16] established first order and second order gradient estimates for positive solutions of the heat equations under general Riemannian-geometric flows. The list of relevant references includes but is not limited to [17-21].

Comparatively, there are less works on Finsler manifolds about gradient estimates of the nonlinear heat equation (1.1). To the best of our knowledge, in [22], Ohta and Sturm derived a Li-Yau gradient estimate as well as parabolic Harnack inequalities on compact Finsler manifolds. In [23], Lakzian derived differential Harnack estimates for positive global solutions to (1.1) under Finsler-Ricci flow. Later, the author and He [24] generalized and corrected Lakzian's results under some curvature constraints. Compared to the Riemannian case, it is harder to get the gradient estimate due to some obstructions. First, the solutions of (1.1) are lack of higher order regularity. Second, $\Delta_m u$ has no definition at the maximum point of u, and thus we cannot use Finsler-Laplacian to adopt maximum principle. Last but not least, in view of nonlinear property of gradient operator, it is difficult to do the calculations. In this paper, we follow the work of Sun [16], and establish some gradient estimates for positive global solutions of (1.1) under Finsler-geometric flow (1.2), which are richer than [16,22-24].

The rest of this paper is organized as follows.

In Section 2, we first briefly review some facts and results about Finsler geometry. In Section 3, we establish space-time gradient estimates for positive global solution of (1.1),