Nonlinear Degenerate Anisotropic Elliptic Equations with Variable Exponents and $L^1$ Data

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Abstract. This paper is devoted to the study of a nonlinear anisotropic elliptic equation with degenerate coercivity, lower order term and $L^1$ datum in appropriate anisotropic variable exponents Sobolev spaces. We obtain the existence of distributional solutions.

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1 Introduction

In this paper we prove the existence of solutions to the nonlinear anisotropic degenerate elliptic equations with variable exponents, of the type

$$
\begin{align*}
-\sum_{i=1}^{N} D_i a_i(x,u,\nabla u) + g(x,u,\nabla u) &= f, & \text{in } \Omega, \\
u &= 0, & \text{on } \partial \Omega,
\end{align*}
$$

(1.1)

where $\Omega \subseteq \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary $\partial \Omega$ and the right-hand side $f$ in $L^1(\Omega)$, $D_i u = \frac{\partial u}{\partial x_i}$. We suppose that $a_i: \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$, $i = 1, \ldots, N$ are

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Carathéodory functions such that for almost every \( x \) in \( \Omega \) and for every \((\sigma,\xi)\in\mathbb{R}\times\mathbb{R}^N\) the following assumptions are satisfied for all \( i=1,\ldots,N \)

\[
|a_i(x,\sigma,\xi)| \leq \beta \left( |k(x)| + |\sigma|^\overline{p}(x) + \sum_{j=1}^{N} |\xi_j|^{p_j(x)} \right)^{1 - \frac{1}{\overline{p}(x)}},
\]

\[
\sum_{i=1}^{N} (a_i(x,\sigma,\xi) - a_i(x,\sigma,\eta)) (\xi_i - \eta_i) > 0, \quad \forall \xi \neq \eta, \quad \forall x \in \Omega.
\]

\[
\sum_{i=1}^{N} a_i(x,\sigma,\xi) \xi_i \geq \alpha \sum_{i=1}^{N} |\xi_i|^{p_i(x)} (1 + |\sigma|)^{\gamma_i(x)},
\]

where \( \beta > 0, \alpha > 0, \) and \( k \in L^1(\Omega), \gamma_i: \Omega \to \mathbb{R}^+, \ p_i: \Omega \to (1, +\infty) \) are continuous functions and \( \overline{p} \) is such that

\[
\frac{1}{\overline{p}(\cdot)} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i(\cdot)}.
\]

We introduce the function

\[
\overline{p}^+(x) = \begin{cases} \frac{N\overline{p}(x)}{N - \overline{p}(x)} & \text{if } \overline{p}(x) < N, \\ +\infty & \text{if } \overline{p}(x) \geq N. \end{cases}
\]

The nonlinear term \( g: \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \) is a Carathéodory function such that for a.e. \( x \in \Omega \) and all \((\sigma,\xi)\in\mathbb{R}\times\mathbb{R}^N\), we have

\[
|g(x,\sigma,\xi)| \leq b(|\sigma|) \left( c(x) + \sum_{i=1}^{N} |\xi_i|^{p_i(x)} \right),
\]

\[
g(x,\sigma,\xi) \cdot \sigma \geq 0,
\]

where \( b: \mathbb{R}^+ \to \mathbb{R}^+ \) is a continuous and increasing function with finite values, \( c \in L^1(\Omega) \) and \( \exists \rho > 0 \) such that:

\[
|g(x,\sigma,\xi)| \geq \rho \left( \sum_{i=1}^{N} |\xi_i|^{p_i(x)} \right), \quad \forall \sigma \text{ such that } |\sigma| > \rho.
\]

In [1], the authors obtain the existence of renormalized and entropy solutions for the nonlinear elliptic equation with degenerate coercivity of the type

\[-\text{div}[a(x,u)|\nabla u|^{p(x)-2}\nabla u] + g(x,u) = f \in L^1(\Omega).\]

For \( g \equiv 0 \) and \( f \in L^{m(\cdot)}(\Omega), \) with \( m(x) \geq m_- \geq 1, \) equation of the from (1.1) have been widely studied in [2], where the authors obtain some existence and regularity results for the solutions. If \( g \equiv |u|^{s(x)-1}u, \)

\[
a_i(x,u,\nabla u) = \frac{|D_i u|^{p_i(x)-2}D_i u}{(1 + |u|)^{\gamma_i(x)}}.
\]