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Fast Algorithm Based on TT-M FE Method for Allen-Cahn Equation

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Abstract. A fast time two-mesh finite element algorithm using coarse and fine meshes is applied to the nonlinear Allen-Cahn equation. The stability and convergence of the method are studied and detailed error estimates are provided. Numerical examples confirm the theoretical results. Traditional Galerkin finite element and time two-mesh finite element methods are compared with respect to CPU time, accuracy and coarsening processing. Numerical experiments show the efficiency and effectiveness of the fast algorithm proposed.

AMS subject classifications: 35Q30, 74S05

Key words: Fast algorithm, time two-mesh finite element method, Allen-Cahn equation, stability, convergence.

1. Introduction

The Allen-Cahn equation is used to model phase separation [1], microstructure evolution [5, 34], dendritic crystal growth [29], multiphase incompressible flows [24, 27], the impact of a droplet on a solid surface [4], image inpainting [2, 9], motion by meaning curvature flow [3, 11], crystal growth [6, 10, 15, 16, 26, 33], and phase-field modeling of tumor growth [28].

Numerical methods are an important tool to study the dynamics of the systems described by the Allen-Cahn equation, but its discretisation meets difficulties because of the presence of the second-order differential operator. Nevertheless, a number of efficient numerical approaches developed recently, includes finite difference schemes inheriting theoretical energy stability [31], finite difference and spectral methods on rectangular regions [7, 12, 17, 18, 21, 30] and general domains of complex geometries [8], discontinuous Galerkin method [14], adaptive mesh refinement [25] and moving meshes method [13]. Lee and Kim [20] discussed an efficient and accurate numerical algorithm based on an operator splitting technique combined with a linear geometric multigrid method. Layton [19]

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introduced two-level methods, which first derive approximate solutions of nonlinear equations in the coarse-level subspaces and then in the fine-level ones. Two-level methods achieve a better accuracy with smaller CPU time. Recently, Liu *et al.* [23] proposed a fast TT-M FE algorithm for time fractional water wave model. It is developed to deal with the time-consuming issue of nonlinear iterations used in standard nonlinear Galerkin finite element (FE) methods.

The time two-mesh (TT-M) finite element algorithm consists of three main steps:

1. Solving a nonlinear FE system on a coarse time mesh by the Newton iterative method.
2. Interpolating solutions on time fine mesh by using the numerical solutions obtained in the first step.
3. Solving the linearised FE system by the interpolation solution on a fine time mesh.

The TT-M FE algorithm [23] is accurate and saves computation time. Yin [35] applied this algorithm to the nonlinear space fractional Allen-Cahn equation.

Here we deal with the Allen-Cahn equation

$$\begin{aligned} &u_t - \varepsilon^2 \Delta u + f(u) = 0, && (z,t) \in \Omega \times (0,T], \\ &u(z,t) = 0, && (z,t) \in \partial\Omega \times (0,T], \\ &u(z,0) = u_0(z), && z \in \Omega, \end{aligned} \tag{1.1}$$

where $\Omega \in \mathbb{R}^2$, $u_t = \partial u/\partial t$, ε is a given parameter, and the function f is usually specified to be $u^3 - u$. Let us recall that u is labelled as the phase-variable and reflects the concentration of one of two substances in a mixture.

The main goal of this work is to accelerate numerical calculations and here we follow the ideas of [23]. Besides, the time derivative is approximated by a second-order scheme [22, 32]. The fast algorithm used here can be extended to the nonlinear Cahn-Hilliard equation and will be published elsewhere.

The rest of the paper is organised as follows. In Section 2 we recall some auxiliary results. Section 3 describes a fast TT-M FE algorithm with the second-order θ scheme. Section 4 deals with the stability of the scheme proposed. In Section 5 we analyse the errors of the method. Section 6 presents the results of numerical experiments and our conclusions are in Section 7.

2. Theoretical Preparations

Let Du refer to the first order derivative of u with respect to x. We set

$$(u,v) := \int_\Omega u(x)v(x)dx, \qquad \|u\| := \|u\|_{L^2(\Omega)},$$
$$|u|_{H^1} := \left(\int_\Omega |Du|^2 dx\right)^{1/2}, \quad \|u\|_{H^1} := \left(\int_\Omega |u|^2 dx + \int_\Omega |Du|^2 dx\right)^{1/2}.$$