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REVIEW ARTICLE

Review of Entropy Stable Discontinuous Galerkin Methods for Systems of Conservation Laws on Unstructured Simplex Meshes

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Abstract. In this paper, we will build a roadmap for the growing literature of high order quadrature-based entropy stable discontinuous Galerkin (DG) methods, trying to elucidate the motivations and emphasize the contributions. Compared to the classic DG method which is only provably stable for the square entropy, these DG methods can be tailored to satisfy an arbitrary given entropy inequality, and do not require exact integration. The methodology is within the summation-by-parts (SBP) paradigm, such that the discrete operators collocated at the quadrature points should satisfy the SBP property. The construction is relatively easy for quadrature rules with collocated surface nodes. We use the flux differencing technique to ensure entropy balance within elements, and the simultaneous approximation terms (SATs) to produce entropy dissipation on element interfaces. The further extension to general quadrature rules is achieved through careful modifications of SATs.

AMS subject classifications: 65M12, 65M60

Key words: System of conservation laws, entropy stability, discontinuous Galerkin method, summation-by-parts.

1 Introduction

Systems of conservation laws describe the phenomena that the production of a conserved quantity in any domain is balanced by a flux through the boundary [22]. Entropy inequalities, which help to single out the "physically relevant" solution, are crucial to the well-posedness of conservation laws. Therefore, when designing numerical methods, we hope that entropy inequalities are satisfied at certain discrete level. Such property is

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called entropy stability. Entropy stability analysis is well-developed for the first order (finite volume) method. The key concepts are Tadmor's entropy conservative fluxes and entropy stable fluxes [81,82]. For high order entropy stable finite volume methods, a major result is the *TeCNO* scheme, proposed by Fjordholm, Mishra and Tadmor [31] as a version of ENO schemes [45]. The authors used high order linear combinations of entropy conservative fluxes in [61], along with the sign property of ENO reconstruction [32].

Discontinuous Galerkin (DG) methods [13–15, 17], due to their local conservation, great parallel efficiency and flexibility for dealing with unstructured meshes, constitute another popular category of high order numerical methods for solving conservation laws. It is well known that the classic DG method satisfies a discrete entropy inequality with respect to the square entropy (i.e., L^2 stability), for scalar conservation laws [58] and symmetric systems [54]. However, the stability result is only valid for the square entropy function. There is no provable stability for problems such that the square function does not define an entropy function. Moreover, we implicitly assume that all integrals in the DG formulation are evaluated exactly. In practice, numerical quadrature rules are usually applied, and the method we actually code up might not be stable. One possible remedy to accomplish entropy stability for an arbitrary given entropy function is to approximate the entropy variables of that entropy function directly (see [2,53,56,87]). This approach is computationally expensive, as we need to solve nonlinear systems at each time step, even for explicit time discretization. Besides, the stability proof still relies on the assumption of exact integration.

Over the past decade, there have been rapid developments on entropy stable quadrature-based DG methods. These DG methods are often characterized in the matrix-vector nodal formulation collocated at the quadrature points [49, 60]. Because of the approximation error induced by quadrature, we no longer have the integration by parts property and the chain rule. The methodology was first developed for the Legendre-Gauss-Lobatto quadrature rule in one space dimension. The corresponding discrete operators (i.e., matrices) were shown to satisfy the summation-by-parts (SBP) property [24, 26, 80], which is the discrete analogue of integration by parts. The distinctive feature of the Gauss-Lobatto quadrature rule is that it contains the two boundary points. Then we make sure that the boundary matrices are diagonal, and neighboring cells can be coupled in a natural way through penalty type terms, usually called simultaneous approximation terms (SATs) in the literature. In order to deal with the loss of chain rule, ad hoc split form methods have been provided for the Burgers equation [36], shallow water equations [39] and Euler equations [38] (for kinetic energy stability). In [4, 5], Carpenter et al. revealed the generic logic behind the splitting procedure by demonstrating the *flux differencing* technique. Flux differencing is essentially high order difference operations on Tadmor's entropy conservative fluxes, and is applicable to any system with any given entropy function.

The one-dimensional Gauss-Lobatto nodal methodology can be easily generalized to multi-dimensional Cartesian meshes through tensor product. In [12], Chen and Shu proposed the entropy stable DG method on unstructured simplex meshes, by intro-