JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **33**, No. 2, pp. 171-192

doi: 10.4208/jpde.v33.n2.6
June 2020

Extremal Functions for Adams Inequalities in Dimension Four

LI Xiaomeng[1,2,*]

[1] *School of Information, Huaibei Normal University, Huaibei 235000, China.*
[2] *Department of Mathematics, Renmin University of China, Beijing 100872, China.*

Received 7 November 2019; Accepted 6 April 2020

Abstract. Let $\Omega\subset\mathbb{R}^4$ be a smooth bounded domain, $W_0^{2,2}(\Omega)$ be the usual Sobolev space. For any positive integer ℓ, $\lambda_\ell(\Omega)$ is the ℓ-th eigenvalue of the bi-Laplacian operator. Define $E_\ell=E_{\lambda_1(\Omega)}\oplus E_{\lambda_2(\Omega)}\oplus\cdots\oplus E_{\lambda_\ell(\Omega)}$, where $E_{\lambda_i(\Omega)}$ is eigenfunction space associated with $\lambda_i(\Omega)$. $E_\ell^\perp$ denotes the orthogonal complement of E_ℓ in $W_0^{2,2}(\Omega)$. For $0\le\alpha<\lambda_{\ell+1}(\Omega)$, we define a norm by $\|u\|_{2,\alpha}^2=\|\Delta u\|_2^2-\alpha\|u\|_2^2$ for $u\in E_\ell^\perp$. In this paper, using the blow-up analysis, we prove the following Adams inequalities

$$\sup_{u\in E_\ell^\perp,\|u\|_{2,\alpha}\le 1}\int_\Omega e^{32\pi^2u^2}\mathrm{d}x<+\infty;$$

moreover, the above supremum can be attained by a function $u_0\in E_\ell^\perp\cap C^4(\overline{\Omega})$ with $\|u_0\|_{2,\alpha}=1$. This result extends that of Yang (J. Differential Equations, 2015), and complements that of Lu and Yang (Adv. Math. 2009) and Nguyen (arXiv: 1701.08249, 2017).

AMS Subject Classifications: 46E35

Chinese Library Classifications: O17

Key Words: Adams inequality; Trudinger-Moser inequality; extremal function; blow-up analysis.

1 Introduction and main result

Trudinger-Moser inequalities play important roles in analysis and geometry. There are two interesting subjects in the study of Trudinger-Moser inequalities: one is what the best constant is, the other is the existence of extremal functions. The research on sharp

*Corresponding author. *Email address:* `xmlimath@ruc.edu.cn` (X. M. Li)

constants was initiated by Yudovich [1], Pohozaev [2] and Trudinger [3]. Later Moser [4] found the best constant: if $\beta \le \beta_0 = n\omega_{n-1}^{1/(n-1)}$, then

$$\sup_{u\in W_0^{1,n}(\Omega),\|\nabla u\|_n=1} \int_\Omega e^{\beta|u|^{n/(n-1)}} \mathrm{d}x < \infty, \tag{1.1}$$

where Ω is an open subset of $\mathbb{R}^n$ $(n\ge 2)$ with finite Lebesgue measure, ω_{n-1} is the measure of the unit sphere in $\mathbb{R}^n$; moreover, if $\beta > \beta_0$, the integrals in (1.1) are still finite, but the supremum is infinite. The sharp constants for higher order derivatives of Moser's inequality was due to Adams [5]. For any fixed positive integer m, let $u \in C_0^m(\Omega)$, the space of functions having m-th continuous derivatives and compact support. To state Adams' result, we use the symbol $\nabla^m u$ to denote the m-th order gradient for u. Precisely

$$\nabla^m u = \begin{cases} \Delta^{\frac{m}{2}} u & \text{when } m \text{ is even,} \\ \nabla \Delta^{\frac{m-1}{2}} u & \text{when } m \text{ is odd,} \end{cases}$$

where ∇ and Δ denote the usual gradient and the Laplacian operators. Adams proved that if $\beta \le \beta(n,m)$ and $0 < m < n$, then

$$\sup_{u\in W_0^{m,\frac{n}{m}}(\Omega),\|\nabla^m u\|_{L^{\frac{n}{m}}(\Omega)}\le 1} \int_\Omega e^{\beta|u|^{n/(n-m)}} \mathrm{d}x \le C_{m,n}|\Omega| \tag{1.2}$$

for some constant $C_{m,n}$, where

$$\beta(n,m) = \begin{cases} \dfrac{n}{\omega_{n-1}} \left[\dfrac{\pi^{n/2} 2^m \Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})} \right]^{\frac{n}{n-m}} & \text{when } m \text{ is odd,} \\ \dfrac{n}{\omega_{n-1}} \left[\dfrac{\pi^{n/2} 2^m \Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})} \right]^{\frac{n}{n-m}} & \text{when } m \text{ is even.} \end{cases}$$

Moreover, $\beta(n,m)$ is the best constant in the sense that if $\beta > \beta(n,m)$, then the supremum in (1.2) is infinite. The manifold version of Adams inequality was obtained by Fontana [6]. Extremal functions for (1.1) were first found by Carleson and Chang [7] when Ω is the unit ball in $\mathbb{R}^n$. This result was then extended by Flucher [8] to a general domain $\Omega \subset \mathbb{R}^2$, and by Lin [9] to a bounded smooth domain $\Omega \subset \mathbb{R}^n$ $(n \ge 2)$.

In 2004, it was proved by Adimurthi and Druet [10] that for any α, $0 \le \alpha < \lambda_1(\Omega)$, there holds

$$\sup_{u\in W_0^{1,2}(\Omega),\|\nabla u\|_2\le 1} \int_\Omega e^{4\pi u^2(1+\alpha\|u\|_2^2)} \mathrm{d}x < +\infty \tag{1.3}$$

and the supremum is infinit for $\alpha \ge \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the first eigenvalue of the Laplacian operator with respect to Dirichlet boundary condition. The inequality (1.3)