JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **33**, No. 2, pp. 119-142

doi: 10.4208/jpde.v33.n2.3
June 2020

On a Quasilinear Degenerate Parabolic Equation from Prandtl Boundary Layer Theory

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Received 22 November 2019; Accepted 6 March 2020

Abstract. The equation arising from Prandtl boundary layer theory

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x_i}\left(a(u,x,t)\frac{\partial u}{\partial x_i}\right)-f_i(x)D_iu+c(x,t)u=g(x,t)$$

is considered. The existence of the entropy solution can be proved by BV estimate method. The interesting problem is that, since $a(\cdot,x,t)$ may be degenerate on the boundary, the usual boundary value condition may be overdetermined. Accordingly, only dependent on a partial boundary value condition, the stability of solutions can be expected. This expectation is turned to reality by Kružkov's bi-variables method, a reasonable partial boundary value condition matching up with the equation is found first time. Moreover, if $a_{x_i}(\cdot,x,t)|_{x\in\partial\Omega}=a(\cdot,x,t)|_{x\in\partial\Omega}=0$ and $f_i(x)|_{x\in\partial\Omega}=0$, the stability can be proved even without any boundary value condition.

AMS Subject Classifications: 35K65, 35L65, 35R35

Chinese Library Classifications: O175.29

Key Words: Prandtl boundary layer theory; entropy solution; Kružkov's bi-variables method; partial boundary value condition; stability.

1 Introduction

The initial-boundary value problem of the quasilinear degenerate parabolic equation

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x_i}\left(a(u,x,t)\frac{\partial u}{\partial x_i}\right)-f_i(x)D_iu+c(x,t)u=g(x,t),\quad (x,t)\in\Omega\times(0,T), \tag{1.1}$$

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$$u(x,0)=u_0(x), \qquad x\in\Omega, \tag{1.2}$$
$$u(x,t)=0, \qquad x\in\partial\Omega, \tag{1.3}$$

is considered in this paper, where $a(u,x,t)\geq 0$, $\Omega\subset\mathbb{R}^N$ is a appropriately smooth open domain, $D_i=\frac{\partial}{\partial x_i}$, the double indices of i represent the summation from 1 to N as usual.

Equation (1.1) arises from the boundary layer theory [1] etc. As the simplification of the Navier-Stokes equation, the Prandtl boundary layer equation describes the motion of a fluid with small viscosity about a solid body in a thin layer which is formed near its surface owing to the adhesion of the viscous fluid to the solid surface. In particular, we consider the motion of a fluid occupying a two dimensional region is characterized by the velocity vector $V=(u,v)$, where u,v are the projections of V onto the coordinate axes x,y, respectively, assume that the density of the fluid ρ is equal to 1, then the Prandtl boundary layer equation for a non-stationary boundary layer arising in an axially symmetric incompressible flow past a solid body has the form [1]

$$\begin{cases} u_t+uu_x+vu_y=\nu u_{yy}-p_x, \\ u_x+v_y=0, \\ u(0,x,y)=u_0(x,y), \quad u(t,0,y)=u_1(t,y), \\ u(t,x,0)=0, \quad v(t,x,0)=v_0(t,x), \\ \lim_{y\to\infty}u(t,x,y)=U(t,x), \end{cases}$$

in a domain $D=\{0<t<T,0<x<X,0<y<\infty\}$, where $\nu=const>0$ is the viscosity coefficient of the incompressible fluid, $u_0>0,u_1>0$ for $y>0$, $u_{0y}>0,u_{1y}>0$ for $y\geq 0$, where, $p=p(t,x)$ is the pressure, $U=U(t,x)$ is the velocity at the outer edge of the boundary layer which satisfies

$$U_t+UU_x=-p_x(t,x), \quad U(t,x)>0.$$

By the well-known Crocco transform,

$$\tau=t, \quad \xi=x, \quad \eta=u(t,x,y), \quad w(\tau,\xi,\eta)=u_y,$$

we can show that $u_y=w$ satisfies the following nonlinear equation

$$w_\tau=\nu w^2 w_{\eta\eta}-\eta w_\xi+p_x w_\eta. \tag{1.4}$$

By a linearized method, Oleinik had shown that there is a local classical solution to this equation [2]. Although there are some important papers to studied the global solutions of the Prandtl boundary layer equation [3-8], the related problems are far from being solved. For example, the compatibility problem between Navier-Stokes equation and Prandtl boundary layer equation. For another example, whether there is a global solution of equation (1.4) and whether this global solution can be deduced a global weak solution of the Prandtl boundary layer equation by the inverse transform of Crocco transform ? In fact, if the domain is not the $N-$dimmensional cube, whether the inverse transform