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Multiple Solutions for a Fractional p-Laplacian Equation with Concave Nonlinearities

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Abstract. We investigate a fractional p-Laplacian equation with right-hand-side nonlinearity which exhibits $(p-1)$ -sublinear term of the form $\lambda|u|^{q-2}, q<p$ (concave term), and a continuous term $f(x,u)$ which is respectively $(p-1)$-superlinear or asymptotically $(p-1)$-linear at infinity. Some existence results for multiple nontrivial solutions are established by using variational methods combined with the Morse theory.

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1 Introduction

For $p\in(1,\infty)$, $s\in(0,1)$ and smooth functions u, define

$$(-\Delta)_p^s u(x)=2\lim_{\epsilon\searrow 0}\int_{\mathbb{R}^N\setminus B_\epsilon(x)}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\mathrm{d}y,\qquad x\in\mathbb{R}^N.$$

This definition is consistent, up to a normalization constant depending on N and s, with the usual definition of the linear fractional Laplacian $(-\Delta)^s$ when $p=2$. In recent years, many authors have focused on the study of these non-local operators of elliptic type, not only the pure mathematical research but also concrete real-world applications, see [1–5] and the references therein. For an elementary introduction of the properties of the fractional Sobolev space, we refer the interested reader to [6]. There is currently a rapidly growing literature on problems involving these nonlocal operators. In particular, the

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eigenvalue problems associated with $(-\Delta)_p^s u$ was studied in [7, 8], regularity theory in [9, 10], and existence theory in the subcritical polynomial case in [11–15].

Following the direction, in this paper we consider the problem

$$\begin{cases} (-\Delta)_p^s u(x) = \lambda |u|^{q-2}u + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where Ω be bounded domain in $\mathbb{R}^N$ with smooth boundary, $1<q<p$, λ is a real parameter and $f \in C(\bar{\Omega}\times\mathbb{R},\mathbb{R})$. Here, we briefly introduce the variation formulation of problem (1.1). Let $\Omega \subset \mathbb{R}^N$ be stated as above, and for all $1 \le r \le \infty$ denote by $|\cdot|_r$ the norm of $L^r(\Omega)$. Moreover, let $0<s<1<p<\infty$ be real numbers, and the fractional critical exponent be defined as $p^* = \frac{Np}{N-sp}$ if $sp<N$ and $p^*=\infty$ if $sp \ge N$. The Gagliardo seminorm is defined for all measurable function $u:\mathbb{R}^N \to \mathbb{R}$ by

$$[u]_{s,p} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}}.$$

Define the fractional Sobolev space

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : u \text{ measurable}, [u]_{s,p} < \infty\},$$

endowed with the norm

$$\|u\|_{s,p} = \left(\|u\|_p^p + [u]_{s,p}^p\right)^{\frac{1}{p}}.$$

Our problem is set in the closed linear subspace

$$X = \{u \in W^{s,p}(\mathbb{R}^N) : u(x) = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega\},$$

which can be equivalently renormed by setting $\|\cdot\| = [\cdot]_{s,p}$ (see Theorem 7.1 in [6]). We know that $(X, \|\cdot\|)$ is a uniformly convex Banach space and that the embedding $X \hookrightarrow L^r(\Omega)$ is continuous for all $1 \le r \le p^*$ and compact for all $1 \le r < p^*$ (see Theorem 6.5, 7.1 in [6]). The dual space of $(X, \|\cdot\|)$ is denoted by $(X^*, \|\cdot\|_*)$. The $(p-1)$-homogeneous nonlinear operator $A: X \to X^*$ is defined for all $u, \varphi \in X$ by

$$\langle A(u), \varphi \rangle = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y.$$

Thus, a weak solution of problem (1.1) is a function $u \in X$ such that

$$\langle A(u), \varphi \rangle = \lambda \int_\Omega |u|^{q-2} u\varphi \mathrm{d}x + \int_\Omega f(x,u)\varphi \mathrm{d}x \tag{1.2}$$

for all $\varphi \in X$. Since X is uniformly convex, by Proposition 1.3 in [16], A satisfies the following compactness condition: If (u_n) is a sequence in X such that $u_n \rightharpoonup u$ in X and $\langle A(u_n), u_n - u \rangle \to 0$, then $u_n \to u$ in X.