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DEVELOPABLE SURFACE PATCHES BOUNDED BY NURBS CURVES*

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Abstract

In this paper we construct developable surface patches which are bounded by two rational or NURBS curves, though the resulting patch is not a rational or NURBS surface in general. This is accomplished by reparameterizing one of the boundary curves. The reparameterization function is the solution of an algebraic equation. For the relevant case of cubic or cubic spline curves, this equation is quartic at most, quadratic if the curves are Bézier or splines and lie on parallel planes, and hence it may be solved either by standard analytical or numerical methods.

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Key words: NURBS, Bézier, Rational, spline, NURBS. Developable surfaces.

1. Introduction

Developable surfaces play an important role in differential geometry as surfaces with vanishing Gaussian curvature. From the point of view of intrinsic geometry, developable surfaces cannot be distinguished from the plane. Only when they are embedded in three-dimensional space, different surfaces arise. The embedding of the planar surface in space has to preserve lengths and angles between curves. Metric properties are not altered and hence the planar surface may be cut or folded, but not stretched or deformed.

On the other hand, these geometrical properties are of relevance for industry. In textile design one starts with a planar piece of cloth to produce garments and their quality improves if the cloth is not stretched. In naval industry one has to adapt planar sheets of steel to the form of the hull of a vessel. This can be done with a folding machine if the result is a developable surface, avoiding the application of heat and reducing the costs. They are also useful for modeling pages of a book [1] for 3D reconstruction and they can also be found in architectural constructions [2].

The main problem for addressing developable surfaces in Geometrical Design is that the null Gaussian curvature requirement takes the form of a non-linear equation when expressed in terms of the vertices of the control net of the surface.

This issue has been handled in several ways. A thorough review may be found in [3]. In [4] rational Bézier surfaces are addressed and the null Gaussian curvature condition is solved explicitly for low degrees. C^2-spline developable surfaces are constructed in [5]. Another restriction is considering boundary curves for the developable surface on parallel planes as in [6] and [7]. A different point of view relies on solving the null Gaussian curvature in the dual space of planes [8–10].

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Concerning the applications in industry, quasi-developable surfaces are constructed in [11] and [12]. In [13] developable surfaces for designing ship hulls are constructed by graphical methods. Developable surfaces can also be approximated with spline cones as in [14]. A different and successful approach for approximate developable surfaces bounded by polylines, grounded on convex hulls, is shown in [15], with examples for garments.

Application of the de Casteljau algorithm has lead to several fruitful approaches as in [16]. In [17] a family of developable surfaces is constructed through a Bézier curve of arbitrary degree. This is useful for solving interpolation problems [18]. These results have been extended to spline curves of arbitrary degree in [19, 20] and to Bézier triangular surfaces [21]. In [22] developable surfaces with several patches linked with G^1-continuity are constructed.

In [23] the non-linear conditions are expressed as quadratic equations and this is used to devise a constraint for interactive modeling.

Finally, in [24] it is shown that the developable surfaces which can be constructed with Aumann's algorithm are the ones with a polynomial edge of regression. This poses an interesting problem. When we interpolate a ruled surface between two parameterized curves, $c(t)$ and $d(t)$, besides the obvious way,

$$b(t,v) = (1-v)c(t) + vd(t), \qquad v \in [0,1],$$

there are other infinite possibilities, depending on the choice of parameterizations for the bounding curves. In this paper we focus on this issue.

The paper is organised as follows: In Section 2 we introduce developable surface patches bounded by rational Bézier curves of arbitrary degree n. We look for the most general solution to this problem by reparameterizing one of the curves. The reparameterization function is shown to satisfy an algebraic equation of degree $2n-2$ at most, or of degree $n-1$ if the bounding curves are polynomial and lie on parallel planes. Examples are provided in Section 3. In Section 4 it is shown how the results can be applied to developable surface patches bounded by NURBS curves. A final section of conclusions is included.

2. Developable Patches Bounded by Rational Curves

We start with two rational curves of degree n, $c(t)$, $d(T)$, $t, T \in [0,1]$ and respective control polygons $\{c_0, \ldots, c_n\}$, $\{d_0, \ldots, d_n\}$ and lists of weights $\{w_0, \ldots, w_n\}$, $\{\omega_0, \ldots, \omega_n\}$. We may think of $T = T(t)$ as a function of t in order to construct a parameterized ruled surface,

$$b(t,v) = (1-v)c(t) + vd(T(t)) = (1-v)c(t) + v\hat{d}(t), \qquad t, v \in [0,1],$$

denoting the reparameterized curve as $\hat{d}(t) := d(T(t))$. We shall denote by a comma the derivative with respect to t and by a dot the derivative with respect to T.

A normal vector $N(t,v)$ to the surface at $b(t,v)$ may be calculated,

$$\begin{aligned} N(t,v) &:= b_t(t,v) \times b_v(t,v) = \Big((1-v)c'(t) + v\hat{d}'(t)\Big) \times \Big(\hat{d}(t) - c(t)\Big) \\ &= (1-v)N(t,0) + vN(t,1), \end{aligned} \tag{2.1}$$

as a barycentric combination of the normal $N(t,0)$ at $c'(t)$ and the normal $N(t,1)$ at $\hat{d}'(t)$.

In the case of developable surfaces [25], $N(t,0)$ and $N(t,1)$ are parallel for all values of t (See Fig. 2.1). In order to avoid singular points for which $N(t,v)$ is a zero vector, we require that