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TWO-STAGE FOURTH-ORDER ACCURATE TIME DISCRETIZATIONS FOR 1D AND 2D SPECIAL RELATIVISTIC HYDRODYNAMICS*

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Abstract

This paper studies the two-stage fourth-order accurate time discretization [J.Q. Li and Z.F. Du, SIAM J. Sci. Comput., 38 (2016)] and its application to the special relativistic hydrodynamical equations. Our analysis reveals that the new two-stage fourth-order accurate time discretizations can be proposed. With the aid of the direct Eulerian GRP (generalized Riemann problem) methods and the analytical resolution of the local “quasi 1D” GRP, the two-stage fourth-order accurate time discretizations are successfully implemented for the 1D and 2D special relativistic hydrodynamical equations. Several numerical experiments demonstrate the performance and accuracy as well as robustness of our schemes.

Mathematics subject classification: 65M08, 65M06, 76M12, 76M20, 76Y05
Key words: Time discretization, Shock-capturing scheme, GRP method, Relativistic hydrodynamics, Hyperbolic conservation laws.

1. Introduction

The relativistic hydrodynamics (RHD) plays the leading role in astrophysics and nuclear physics etc. The RHDs is necessary in situations where the local velocity of the flow is close to the light speed in vacuum, or where the local internal energy density is comparable (or larger) than the local rest mass density of fluid. The paper is concerned with developing higher-order accurate numerical schemes for the 1D and 2D special RHDs. The d-dimensional governing equations of the special RHDs is a first-order quasilinear hyperbolic system. In the laboratory frame, it can be written into the divergence form

$$\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{t}} + \sum_{i=1}^{d} \frac{\partial \boldsymbol{F}_i(\boldsymbol{U})}{\partial x_i} = 0, \tag{1.1}$$

where $d = 1$ or 2, and $\boldsymbol{U}$ and $\boldsymbol{F}_i(\boldsymbol{U})$ denote the conservative vector and the flux in the x_i-direction, respectively, defined by

$$\boldsymbol{U} = (D, \boldsymbol{m}, E)^T, \quad \boldsymbol{F}_i(\boldsymbol{U}) = (Dv_i, v_i\boldsymbol{m} + p\boldsymbol{e}_i, m_i)^T, \quad i = 1, \cdots, d, \tag{1.2}$$

with the mass density $D = \rho W$, the momentum density (row) vector $\boldsymbol{m} = DhW\boldsymbol{v}$, the energy density $E = DhW - p$, and the row vector $\boldsymbol{e}_i$ denoting the i-th row of the unit matrix of size

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2. Here ρ is the rest-mass density, v_i denotes the fluid velocity in the x_i-direction, p is the gas pressure, $W = 1/\sqrt{1-v^2}$ is the Lorentz factor with $v := \left(v_1^2 + \cdots + v_d^2\right)^{1/2}$, h is the specific enthalpy defined by

$$h = 1 + e + \frac{p}{\rho},$$

with units in which the speed of light c is equal to one, and e is the specific internal energy. Throughout the paper, the equation of state (EOS) will be restricted to the Γ-law

$$p = (\Gamma - 1)\rho e, \tag{1.3}$$

where the adiabatic index $\Gamma \in (1, 2]$. The restriction of $\Gamma \leq 2$ is required for the compressibility assumptions and the causality in the theory of relativity (the sound speed does not exceed the speed of light $c = 1$).

The RHD equations (1.1) are highly nonlinear so that their analytical treatment is extremely difficult. Numerical computation has become a major way in studying RHDs. The pioneering numerical work can date back to the Lagrangian finite difference code via artificial viscosity for the spherically symmetric general RHD equations [19, 20]. Multi-dimensional RHD equations were first solved in [26] by using the Eulerian finite difference method with the artificial viscosity technique. Later, modern shock-capturing methods were extended to the RHD (including RMHD) equations. Some representative methods are the HLL (Harten-Lax-van Leer) scheme [6], HLLC (HLLC contact) schemes [12, 21], Riemann solver [2], approximate Riemann solvers based on the local linearization [15, 16], second-order GRP (generalized Riemann problem) schemes [30, 37, 38], third-order GRP scheme [36], locally evolution Galerkin method [29], discontinuous Galerkin (DG) methods [39,40], gas-kinetic schemes (GKS) [4, 23], adaptive mesh refinement methods [1, 13], and moving mesh methods [10, 11] etc. Recently, the higher-order accurate physical-constraints-preserving (PCP) WENO (weighted essentially non-oscillatory) and DG schemes were developed for the special RHD equations [24, 31, 33]. They were built on studying the admissible state set of the special RHDs. The admissible state set and PCP schemes of the special ideal RMHDs were also studied for the first time in [32], where the importance of divergence-free fields was revealed in achieving PCP methods. Those works were successfully extended to the special RHDs with a general equation of state [33, 34] and the general RHDs [28].

In comparison with second-order shock-capturing schemes, the higher-order methods can provide more accurate solutions, but they are less robust and more complicated. For most of the existing higher-order methods, the Runge-Kutta time discretization is usually used to achieve higher order temporal accuracy. For example, a four-stage fourth-order Runge-Kutta method (see e.g. [40]) is used to achieve a fourth-order time accuracy. If each stage of the time discretization needs to call the Riemann solver or the resolution of the local GRP, then the shock-capturing scheme with multi-stage time discretization for (1.1) is very time-consuming. Recently, based on the time-dependent flux function of the GRP, a two-stage fourth-order accurate time discretization was developed for Lax-Wendroff (LW) type flow solvers, particularly applied for the hyperbolic conservation laws [18]. Such two-stage LW time stepping method does also provide an alternative framework for the development of a fourth-order GKS with a second-order flux function [22].

The aim of this paper is to study the two-stage fourth-order accurate time discretization [18] and its application to the special RHD equations (1.1). Based our analysis, the new two-stage fourth-order accurate time discretizations can be proposed. With the aid of the direct