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CONVERGENCE ANALYSIS OF PARAREAL ALGORITHM BASED ON MILSTEIN SCHEME FOR STOCHASTIC DIFFERENTIAL EQUATIONS*

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Abstract

In this paper, we propose a parareal algorithm for stochastic differential equations (SDEs), which proceeds as a two-level temporal parallelizable integrator with the Milstein scheme as the coarse propagator and the exact solution as the fine propagator. The convergence order of the proposed algorithm is analyzed under some regular assumptions. Finally, numerical experiments are dedicated to illustrate the convergence and the convergence order with respect to the iteration number k, which show the efficiency of the proposed method.

Mathematics subject classification: 60H10, 60H35, 65Y05.
Key words: Stochastic differential equations, Parareal algorithm, Convergence, Stochastic Taylor expansion, Milstein scheme.

1. Introduction

The technique of parallel algorithm attracted more and more attention in the past few years due to the computational time and memory issues in the solution of large scale problems. The parallel algorithms in spatial direction is contributed by the domain decomposition method when the system governed by partial differential equations, see [15] and references therein. For the time-dependent problem, there have been developed some time parallel techniques, such as the waveform relaxation methods, the multigrid methods, the diagonalization methods and the domain decomposition methods (see [8] and references therein).

The parareal algorithm, our focus in the sequel, is a two-level temporal parallelizable integrator, which was proposed firstly in [12] and has gone deep into researching in [8] and references therein. The general idea of the parareal algorithm is described through a coarse propagator calculated on a coarse grid with step size ΔT and a fine propagator calculated in parallel on

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each coarse interval with a fine step size $\Delta t = \Delta T/J$, where J can be seen as the number of computer processes. Combining the values of the coarse time grids and the values obtained by the fine propagator performed the parallel, a new approximation is generated by a prediction and correction iteration. Roughly speaking, the fine propagator and the coarse propagator denote the accuracy and efficiency of the parareal algorithm, respectively. In [8] and [12], the authors pointed out that the error caused by the parareal architecture after a few iterations is comparable to the error caused by a global use of the fine propagator without iteration. Since the parareal algorithm was proposed, many efforts have been made to analyze it theoretically and numerically, which verify the effectiveness of the parareal algorithm for a large various of problems, including its convergence [4, 20], its stability [2, 18], the potential of long time simulation [5, 7, 9] and the other applications of the different models and problems [1, 6, 13, 21].

In the stochastic case, there are very few literatures to consider this problem compared to the deterministic case. For example, [16] considered the time simulation of the parareal algorithm of multiscale stochastic chemical kinetics. The authors studied the stability of θ-scheme for a linear SDE in [19]. In [17], the authors adopted a parallel time integration scheme to track the trajectories of noisy nonlinear dynamics systems. Recently, the parareal algorithm applied to stochastic differential equations with conserved quantities was considered in [22], and a parareal algorithm based on exponential θ-scheme was proposed for the stochastic Schrödinger equation with weak damping and additive noise in [10].

Different from the deterministic problems, the main difference when applying the parareal algorithms for SDEs driven by standard Brownian motions, is that the stochastic systems are less regular. One may not get the optimal convergence rate of the parareal algorithm for the stochastic case following the procedure of the deterministic case. The author in [3] considered the parareal algorithm when the explicit Euler scheme is chosen as the coarse propagator and the exact solution as the fine propagator. The optimal rate $\frac{k(\alpha\wedge 3-1)}{2}$ is deduced taking advantages of the independency between the increments of Brownian motions, where α is variant with different drift and diffusion coefficients and $\alpha = 2$ for general coefficients function. Similar idea with [3], we propose a parareal algorithm based on the explicit Milstein scheme as the coarse propagator and the exact solution as the fine propagator in this paper. The motivation of this work is hope to improve the convergence order when using the higher order numerical scheme for the coarse propagator. Based on the form of the Itô Taylor expansion and the tool of stochastic analysis, we obtain the uniform convergence of the proposed parareal algorithm with convergence order k in the sense of mean square when the Milstein scheme as the coarse propagator is order 1. From the result of the convergence order, we find that it can improve the order of the parareal algorithm when using the higher order coarse propagator compared to [3]. Numerical simulations are investigated to verify the effectiveness and convergence of the algorithm for a linear and nonlinear SDEs case.

The rest of this paper is organized as follows. In Section 2, we review the process of the parareal algorithm and stochastic Taylor expansion and then propose the parareal algorithm based on the Milstein scheme. In Section 3, the convergence result in the sense of mean square is analyzed. Two numerical experiments are given in Section 4 to verify our theoretical results.

2. Preliminary

Let $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which is a nondecreasing right continuous family of σ sub-algebra of $\mathcal{F}$, and where $\mathcal{F}_0$ contains all the $\mathbb{P}$-