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A FULLY DISCRETE IMPLICIT-EXPLICIT FINITE ELEMENT METHOD FOR SOLVING THE FITZHUGH-NAGUMO MODEL*

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Abstract

This work develops a fully discrete implicit-explicit finite element scheme for a parabolic-ordinary system with a nonlinear reaction term which is known as the FitzHugh-Nagumo model from physiology. The first-order backward Euler discretization for the time derivative, and an implicit-explicit discretization for the nonlinear reaction term are employed for the model, with a simple linearization technique used to make the process of solving equations more efficient. The stability and convergence of the fully discrete implicit-explicit finite element method are proved, which shows that the FitzHugh-Nagumo model is accurately solved and the trajectory of potential transmission is obtained. The numerical results are also reported to verify the convergence results and the stability of the proposed method.

Mathematics subject classification: 65M12, 65M60.
Key words: Finite element method, Nonlinear reaction term, FitzHugh-Nagumo model, Implicit-explicit scheme, Stability and error estimates.

1. Introduction

The mathematical modeling of cardiac electrical activity has gained a lot attention in medicine and science. It allows reaearchers to better understand biophysical phenomena and contributes to new diagnostic techniques and drug development. In general, existing models can be classified as microscopic or macroscopic scale models. The Grandi-Pasqualini-Bers (GPB) model [6, 11], one of the microscopic scale models, can match experimental data well though discrepancies exist in cellular and tissue levels [10]. At macroscopic scale, there exists two categories. One is the so-called bidomain model [8, 12, 24], which is a system of two nonlinear partial differential equations distinguished from intracellular and extracellular potentials, and coupled with a system of ordinary differential equations representing the ionic currents activity. The other is the simplified mathematical model, the so-called monodomain model [28, 31],

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which only deals with a transmembrane potential, coupled with the same system of ordinary differential equations. The monodomain model has the following form [4]:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (K \nabla u) + I_{ion}(u,v) + I_s, \\ \dfrac{\partial v}{\partial t} = F(u,v), \end{cases} \tag{1.1}$$

where u is a normalized transmembrane potential, v is a time dependent recovery variable, K is the diffusion coefficient, I_s is the current on behalf of the external stimuli and $I_{ion}(u,v)$ is the ionic current across the membranes.

Many methods had been used to solve the coupled system (1.1). Yamada and Nozaki in [36] developed an asymptotic method to solve the FitzHugh-Nagumo model, they found bound states of two or three nerve impulses. Zhang [37] used the element-free Galerkin (EFG) method to solve the system and studied the effects of complex heart geometry, nonuniform fiber orientation, and inhomogeneous materials to electrical propagation. Trangenstein used the operator splitting and mesh refinement method for the FitzHugh-Nagumo model in [34]. And Shuaiby [31] proposed a method which combines a Galerkin finite element method and the operator splitting technique, where the behavior of the excitation and the repolarization phase are shown. In [26], Rahman and Isiam solved the FitzHugh-Nagumo model using the Galerkin finite element method. Rogers and Mcculloch [25] developed a hybrid collocation-Galerkin finite element method for the FitzHugh-Nagumo model and simulated the effects on cardiac impulse propagation. The existence and uniqueness of the solution to system (1.1) were proved in [20, 21]. The theoretical analysis of the semi-discrete space approximation using the finite element method for the reaction-diffusion systems are presented in [19, 30]. In addition, some related works on the implicit-explicit schemes for the Navier-Stokes equations and other high-accuracy numerical methods are given in [13–16, 18].

In this paper, we present a fully discrete implicit-explicit finite element scheme to solve the FitzHugh-Nagumo monodomain model, which satisfies the following equations on a bounded domain Ω with initial and boundary conditions:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (K \nabla u) + G(u)u - v \quad (x,y,t) \in \Omega \times (0,T], \\ \dfrac{\partial v}{\partial t} = \varepsilon(\beta u - \gamma v - \delta) \quad (x,y,t) \in \Omega \times (0,T], \\ u(x,y,0) = \psi(x,y), v(x,y,0) = \varphi(x,y) \quad (x,y) \in \Omega, \\ u(x,y,t) = 0, v(x,y,t) = 0 \quad (x,y,t) \in \partial\Omega \times (0,T], \end{cases} \tag{1.2}$$

where $I_{ion}(u,v) = G(u)u - v, G(u)u = (1-u)(u-a)u, 0 < a < 1$, $I_s = 0$ and $F(u,v) = \varepsilon(\beta u - \gamma v - \delta)$ with $\varepsilon\beta > 0$, $\varepsilon\gamma$, $\varepsilon\delta \geq 0$ in (1.2) for the FitzHugh-Nagumo model.

The remainder of this paper is organized as follows. Section 2 gives the continuous and discrete formulation of system (1.2). In Section 3, the stability and convergence of the implicit-explicit fully discrete finite method are established. The numerical results are given in Section 4. We give a brief summary in the Section 5. Throughout this paper, $(\cdot,\cdot)$ denotes the inner product on $L^2(\Omega)$, and we also give the norm notations: $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_k = \|\cdot\|_{H^k(\Omega)}$, $\|\cdot\|_{0,\mu} = (\int_0^T \|\cdot\|_\mu^2 dt)^{1/2}$.