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THE SHIFTED-INVERSE POWER WEAK GALERKIN METHOD FOR EIGENVALUE PROBLEMS*

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Abstract

This paper proposes and analyzes a new weak Galerkin method for the eigenvalue problem by using the shifted-inverse power technique. A high order lower bound can be obtained at a relatively low cost via the proposed method. The error estimates for both eigenvalue and eigenfunction are provided and asymptotic lower bounds are shown as well under some conditions. Numerical examples are presented to validate the theoretical analysis.

Mathematics subject classification: 65N30, 65N15, 65N25.
Key words: Weak Galerkin finite element method, Eigenvalue problem, Shifted-inverse power method, Lower bound.

1. Introduction

The eigenvalue problems have drawn much attention during the past several decades and have wide applications in physical and industrial fields, such as quantum mechanics, fluid mechanics, stochastic process, structural mechanics. More applications of eigenvalue problems are illustrated in [10] and the references therein.

Many numerical methods have been developed for solving eigenvalue problems, such as finite difference method [18,27], finite element method [2,3], spectral method [29], and discontinuous Galerkin method [9]. However, there are still two difficulties in solving eigenvalue problems. One is that the eigenvalue problem is a semilinear problem and the computational cost is very high. Therefore it is important to design algorithms to reduce the computational complexity. The other difficulty is getting a lower bound of an eigenvalue. Due to the minimum-maximum principle, the conforming finite element approximations always produce upper bound of the exact eigenvalue. If a lower bound is given, then we can get an interval to which the eigenvalue belongs and derive a more accurate approximate eigenvalue. This idea was first explored in [28].

Numerical techniques have been developed to accelerate the computation of the eigenvalue problems. A two-grid method was firstly proposed by Xu in [39] for semi-linear partial differential equations (PDEs). It was soon been applied to nonlinear PDEs [40] and the eigenvalue problems [42]. The two-grid method for nonconforming element methods was first studied

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in [22]. The main idea of the two-grid method is to solve the eigenvalue problem on a coarse grid and a linear problem on a fine grid, instead of solving the eigenvalue problem on the fine grid directly. Meanwhile, the asymptotic convergence rate is maintained as long as fine grid mesh size h and coarse grid mesh size H are chosen properly. For example, for the Laplacian eigenvalue problem, the ratio of mesh sizes of two grids can be $H = \sqrt{h}$, which shall greatly reduces the computation cost. The two-grid method has also been used in many other problems [43,44], and some multigrid methods have also been proposed [8, 17, 38].

Based on the two-grid method, a shifted-inverse power technique was developed [16, 45], which further reduces the computational cost because the coarse grid mesh size can be chosen as $H = \sqrt[4]{h}$. The shifted-inverse power technique can also be combined with other numerical methods, such as multigrid method [7,46] and adaptive algorithm [4], which can solve eigenvalue problems more efficiently. The shifted-inverse power technique usually leads to a nearly singular linear system. In general, this nearly singular linear system actually helps for finding eigenvalues and eigenfunction as discussed in [16]. Moreover, some efficient solvers for nearly singular systems were introduced in [5, 11, 41] and the references therein. In addition, other multigrid methods, which are not based on the shifted-inverse power technique, have been investigated in [24, 38].

On the other hand, since the conforming finite element methods fail to produce a lower bound for the eigenvalues naturally, a variety of non-standard finite element methods have been developed. A posterior analysis was proposed to provide a lower bound [6, 19]. Many non-conforming elements have also been studied for the lower bound problem, such as Wilson's element, EQ_1^{rot} element, and ECR element [21, 23, 28]. Some criterions for non-conforming elements have been studied in [12–14] and some numerical methods of getting both upper and lower bounds have been discussed in [15, 28]. Further exploration of the upper/lower bound problems can be found in [14, 26, 52] and the references therein.

Among the numerous methods above, the weak Galerkin (WG) method is also a candidate for solving the lower bound problem. The weak Galerkin finite element method was proposed by Wang and Ye in [34] and can be applied on polytopal/polyhedra mesh. The key of weak Galerkin method is to employ discontinuous basis functions and use specifically defined weak derivatives to replace the classical derivatives. The weak Galerkin method has been applied to many types of PDEs, such as biharmonic equation [31,32,51], Stokes equation [36,50], Brinkman equation [30, 37, 49], and Maxwell equation [33]. In [47, 48], the weak Galerkin method has been used to solve the Laplacian eigenvalue problems and provide asymptotic lower bounds of arbitrary high order.

In this paper, we combine the shifted-inverse power technique with the weak Galerkin method. The shifted-inverse power technique reduces the computational cost of weak Galerkin method, while the weak Galerkin method provides a lower bound estimate under certain conditions. Therefore, by combining the weak Galerkin method with the shifted-inverse power method, we are able to get a high order lower bound efficiently.

This paper is constructed as follows. In Section 2, the weak Galerkin scheme in the general setting is introduced. Section 3 is devoted to the error analysis for the shifted-inverse power weak Galerkin method. In Section 4, the application of the proposed method to Laplacian and biharmonic eigenvalue problems are analyzed. Numerical experiments are presented in Section 5.