

Remark on the Lifespan of Solutions to 3-D Anisotropic Navier Stokes Equations

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Received 10 February 2020; Accepted 29 February 2020

Abstract. The goal of this article is to provide a lower bound for the lifespan of smooth solutions to 3-D anisotropic incompressible Navier-Stokes system, which in particular extends a similar type of result for the classical 3-D incompressible Navier-Stokes system.

AMS subject classifications: 35Q30, 76D03

Key words: Anisotropic Navier-Stokes system, Littlewood-Paley theory, lifespan.

1 Introduction

In this article, we shall investigate the lifespan for smooth enough solutions to the following 3-D anisotropic incompressible Navier-Stokes system:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta_h u = -\nabla p, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases} \quad (1.1)$$

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where $\Delta_h \stackrel{\text{def}}{=} \partial_1^2 + \partial_2^2$, u designates the velocity of the fluid and p the scalar pressure function which guarantees the divergence free condition of the velocity field.

Systems of this type appear in geophysical fluid dynamics (see for instance [5, 16]). In fact, meteorologists often modelize turbulent diffusion by putting a viscosity of the form: $-\mu_h \Delta_h - \mu_3 \partial_3^2$, where μ_h and μ_3 are empirical constants, and μ_3 is usually much smaller than μ_h . We refer to the book of Pedlovsky [16], Chap. 4 for a complete discussion about this model.

We remark that for the classical Navier-Stokes system (NS), which corresponds to (1.1) with Δ_h there being replaced by $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$, Leray [12] proved the global existence of weak solutions to (NS) in 1934. Yet the uniqueness and regularity to such weak solutions are still open. In [6], Chemin and Gallagher showed that: let u_0 be a regular solenoidal vector field, then the classical Navier-Stokes system (NS) has a unique regular solution on $[0, T]$. Let $T^*(u_0)$ be the maximal time of existence of this regular solution. Then for any $\gamma \in (0, 1/4)$, a positive constant C_γ exists so that

$$T^*(u_0) \geq C_\gamma \|u_0\|_{\dot{H}^{\frac{1}{2}+2\gamma}}^{-\frac{1}{\gamma}}. \quad (1.2)$$

In the special case when $\gamma = \frac{1}{4}$, this type of result goes back to the seminal work of Leray [12]. Lately the same type of result has been proved for 3-D inhomogeneous incompressible Navier-Stokes system in [17] by the second author.

Considering that the system (1.1) has only horizontal dissipation, it is reasonable to use anisotropic Sobolev space defined as follows:

Definition 1.1. For any (s, s') in \mathbb{R}^2 , the anisotropic Sobolev space $\dot{H}^{s, s'}(\mathbb{R}^3)$ denotes the space of homogeneous tempered distribution a such that

$$\|a\|_{\dot{H}^{s, s'}}^2 \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} |\tilde{\zeta}_h|^{2s} |\tilde{\zeta}_3|^{2s'} |\widehat{a}(\tilde{\zeta})|^2 d\tilde{\zeta} < \infty \quad \text{with} \quad \tilde{\zeta}_h = (\tilde{\zeta}_1, \tilde{\zeta}_2).$$

Mathematically, Chemin et al. [4] first studied the system (1.1). In particular, Chemin et al. [4] and Iftimie [11] proved that (1.1) is locally well-posed with initial data in $L^2 \cap \dot{H}^{0, \frac{1}{2}+\varepsilon}$ for some $\varepsilon > 0$, and is globally well-posed if in addition

$$\|u_0\|_{L^2}^\varepsilon \|u_0\|_{\dot{H}^{0, \frac{1}{2}+\varepsilon}}^{1-\varepsilon} \leq c \quad (1.3)$$

for some sufficiently small constant c .

Paicu [14] improved the well-posedness result in [4, 11] to be the critical case, namely, with initial data in the critical anisotropic Besov space, which basically