A TWO-GRID METHOD FOR THE $C^0$ INTERIOR PENALTY DISCRETIZATION OF THE MONGE-AMPÈRE EQUATION

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Abstract

The purpose of this paper is to analyze an efficient method for the solution of the nonlinear system resulting from the discretization of the elliptic Monge-Ampère equation by a $C^0$ interior penalty method with Lagrange finite elements. We consider the two-grid method for nonlinear equations which consists in solving the discrete nonlinear system on a coarse mesh and using that solution as initial guess for one iteration of Newton’s method on a finer mesh. Thus both steps are inexpensive. We give quasi-optimal $W^{1,\infty}$ error estimates for the discretization and estimate the difference between the interior penalty solution and the two-grid numerical solution. Numerical experiments confirm the computational efficiency of the approach compared to Newton’s method on the fine mesh.

Key words: Two-grid discretization, Interior penalty method, Finite element, Monge-Ampère.

1. Introduction

In this paper, we prove the convergence of a two grid method for solving the nonlinear system resulting from the discretization of the elliptic Monge-Ampère equation

$$\det(D^2u) = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial \Omega, \quad (1.1)$$

with a version of the $C^0$ interior penalty discretization proposed in [5]. The domain $\Omega$ is assumed to be a convex polygonal domain of $\mathbb{R}^2$ and (1.1) is assumed to have a strictly convex smooth solution $u \in C^{k+1}(\Omega)$ for an integer $k \geq 3$. The function $f \in C^{k-1}(\overline{\Omega})$ is given and satisfies $f \geq c_0$ for a constant $c_0 > 0$ and the function $g \in C(\partial \Omega)$ is also given and assumed to extend to a $C^{k+1}(\Omega)$ function $G$. In (1.1), $D^2u = (\partial^2 u/\partial x_i \partial x_j)_{i,j=1,2}$ is the Hessian matrix of $u$ and $\det$ denotes the determinant operator. Let $V_h \subset H^1(\Omega)$ denote the Lagrange finite element space of degree $k \geq 3$. Let $Dv$ denote the gradient of the function $v$. Recall that
cof $D^2v$ denotes the matrix of cofactors of $D^2v$. The $C^0$ interior penalty discretization can be written in abstract form as: find $u_h \in V_h$ such that $u_h = g_h$ on $\partial \Omega$ and

$$A(u_h, \phi) = 0, \quad \forall \phi \in V_h \cap H^1_0(\Omega).$$

Here $g_h$ denotes the canonical interpolant in $V_h$ of a continuous extension of $g$ and $A$ is defined in (3.1) below. The discretization has the property that if we denote by $A'(u; v, \phi)$ the Fréchet derivative evaluated at $u$ of the mapping $v \to A(v, \phi)$, then

$$A'(u; v, \phi) = \int_{\Omega} ((\text{cof} \ D^2u) Dv) \cdot D\phi \, dx,$$  \hspace{1cm} (1.2)

which gives the weak form of a standard linear elliptic operator. We exploit this property to give quasi-optimal $W^{1,\infty}$ error estimates, and the convergence of a two-grid numerical scheme for solving the discrete nonlinear system. Numerical experiments confirm the computational efficiency of the two-grid method compared to Newton’s method on the fine mesh. Two-grid methods were initially analyzed in [13] for quasi-linear problems, and (1.1) is a fully nonlinear equation. The numerical results in [11] used a two-grid method.

Monge-Ampère type equations with smooth solutions on polygonal domains appear in many problems of practical interest. For example they appear in the study of von Kármán model for plate buckling [6]. In addition, for meteorological applications for which other differential operators are discretized with a finite element method, it would be advantageous to use a finite element discretization for the Monge-Ampère operator as well. It is known that when $\Omega$ is strictly convex with a smooth boundary, and with our smoothness assumptions on $f$ and $g$, (1.1) has a smooth solution. There are several discretizations for smooth solutions of (1.1). Provably convergent schemes for non smooth solutions can be used for smooth solutions as well. However the latter have a low order of approximation for smooth solutions. We refer to [8] for example for a review. Because the interior penalty term involves the cofactor matrix of the Hessian, it is very likely that the method proposed in [5] is suitable only for smooth solutions. It does not seem possible to put it in the framework of approximation by smooth solutions proposed in [2], where the right hand side of (1.1) is viewed as a measure.

There has been no previous study of multilevel methods for finite element discretizations of (1.1). A key tool in the proof of convergence of the two-grid method is a $W^{1,\infty}$ norm error estimate for $k \geq 3$. Such estimates were obtained in [10] for quadratic and higher order elements on a smooth domain. But the proof therein relies on an elliptic regularity property of the linearized problem [10, (2.21)]. Unless the domain is a rectangle, we do not expect such an elliptic regularity property to hold for general polygonal domains considered in this paper.

With the quasi-optimal $W^{1,\infty}$ error estimates we obtain a new proof of the optimal $H^1$ estimates obtained in [5]. Although these estimates are not new, we include nevertheless the proof since its ideas are also used in the proof of the convergence of the two-grid method. The version of the $C^0$ interior penalty discretization proposed in [5] we consider, consists in imposing the boundary condition through interpolation, instead of weakly with a penalty term. In this context, as expected, the proof of the $H^1$ estimates is simpler than the one given in [5]. In particular, no mesh-dependent norms are used.

The two-grid method consists in solving (1.1) on a coarse mesh of size $H$ and using that solution as initial guess for one iteration of Newton’s method on a finer mesh of size $h$ with $H = h^\lambda, 0 < \lambda < 1$. Thus both steps are inexpensive. We prove that the convergence rate does