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HIGH ORDER FINITE DIFFERENCE/SPECTRAL METHODS TO A WATER WAVE MODEL WITH NONLOCAL VISCOSITY*

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Abstract

In this paper, efficient numerical scheme is proposed for solving the water wave model with nonlocal viscous term that describe the propagation of surface water wave. By using the Caputo fractional derivative definition to approximate the nonlocal fractional operator, finite difference method in time and spectral method in space are constructed for the considered model. The proposed method employs known 5/2 order scheme for fractional derivative and a mixed linearization for the nonlinear term. The analysis shows that the proposed numerical scheme is unconditionally stable and error estimates are provided to predict that the second order backward differentiation plus 5/2 order scheme converges with order 2 in time, and spectral accuracy in space. Several numerical results are provided to verify the efficiency and accuracy of our theoretical claims. Finally, the decay rate of solutions are investigated.

Mathematics subject classification: 35R11, 65L12, 76M22, 65L20.
Key words: Water waves, Nonlocal viscosity, Finite difference, Spectral method, Convergence order, Decay rate.

1. Introduction

The study of shallow water wave propagation plays an important role in many branches of science and technology and has a long history in research fields. Under the assumption of small wave amplitude and large wave length, the KdV (Korteweg-de Vries) equation [1] was originally derived for water waves and it is similarly justifiable as a model for long waves in many other physical systems. The improvement of KdV equation for modeling long surface gravity waves of small amplitude is known as BBM (Benjamin-Bona-Mahony) equation [2]. It had been shown that the BBM equation possesses stability and uniqueness of solutions [3]. This contrasts with KdV equation which is unstable in its high wave number components.

Modeling the effect of viscosity on the water waves is a challenging issue and much research on this subject has been carried out during last decade. In [4], Dutykh and Dias have introduced a system which models water waves in a fluid layer of finite depth under the influence of viscous effects. The model is a generalization of the ones introduced by Liu and Orfila [5] and Bona et al. [6]: it contains the same nonlocal viscosity term as in [5] and it has the flexibility of taking the horizontal velocity at various water levels as in [6]. The linear analysis of the nonlocal term shows that it has both dispersive and dissipative effects. This phenomena has been first pointed out by Kakutani and Matsuuchi [7], whose model involves a nonlocal term in space and

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is equivalent to the model (1.1) [8] for long waves. The Kakutani and Matsuuchi model itself is similar to the famous Ott-Sudan [9] model, supplemented by a nonlocal dispersive term. A one way reduction of the model (1.1) [8] was addressed in [10].

Computing the decay rate and effect of viscosity for solutions of that type of problem is also a challenging issue. The pioneer work is due to C. Amic et al. [11] where the authors handle the decay rate of solutions solutions of KdV-Burgers solutions for any initial data. Bona et al. [12] presented an asymptotic form which renders explicit the relative strengths of the dissipative and dispersive effects in the solutions that decay to zero as time tends to infinity.

Many different numerical methods have been developed in the literature for the computation of water waves models; e.g. [13–19]. But there are not many works on numerical methods for the water waves models involving the fractional terms. A brief overview of the numerical studies for the water waves models involving the fractional derivatives are given below. Chen et al. [8] were concerned with computing both theoretically and numerically the decay rate of solutions to a water wave model with a nonlocal viscous dispersive term. A method involving a semi-implicit scheme for the time discretisation and Fourier approach in space was proposed to compute the decay rates. Goubet and Warnault [20] discussed a linear viscous asymptotic model for water waves and the decay rate of solutions towards the equilibrium. Dumont and Duval [21] investigated the asymptotical decay rate of the solutions, the role of the nonlocal viscous term, the geometric dispersion and the nonlinearity in two asymptotic models by using Gear scheme. Zhang and Xu [22] constructed finite difference/spectral approximations to a water wave model with nonlocal viscous term. The proposed schemes are unconditionally stable and the combined schemes converges with order of $\mathcal{O}(\Delta t^{3/2} + N^{1-m})$. The suggested methods are used to investigate the asymptotic decay rate of the solutions of the water wave equation as well as the impact of different terms on this decay rate. Jennings [23] discussed some efficient numerical methods for the fractional water wave equation in unbounded domains with nonreflecting boundaries. The fractional term in the model [24] is described by the Riemann-Liouville half derivative instead of that of Caputo. The local and global existence of solutions to the model problem is proved for the limiting case using a fixed point theorem. Recently, a semi-implicit spectral defect correction method is constructed for a nonlocal Kakutani-Matsuuchi model [25]. More recently, efficient numerical schemes are proposed for solving the fractional water wave models that describe the propagation of surface water wave [26]. The analysis shows that the proposed numerical schemes are unconditionally stable with second order accuracy for both temporal and spatial discretizations.

In this paper, we consider the efficient numerical scheme for solving the nonlocal water wave model in the BBM form:

$$\partial_t u + \partial_x u - \beta \partial_{xxt} u + \frac{\nu^{1/2}}{\Gamma(1/2)} \int_0^t \frac{\partial_s u(s)}{(t-s)^{1/2}} ds + \gamma u \partial_x u - \alpha \partial_{xx} u = 0, \tag{1.1}$$

where u is the horizontal velocity of the fluid. The usual diffusion is $-\alpha u_{xx}$, while $-\beta u_{xxt}$ is the geometric dispersion and $\frac{\nu^{1/2}}{\Gamma(1/2)} \int_0^t \frac{\partial_s u(s)}{(t-s)^{1/2}} ds$ stands for the nonlocal diffusive-dispersive term and model the viscosity. Here α, β, γ and ν are non negative parameters dedicated to balance or unbalance the effects of viscosity and dispersion versus the nonlinear effects. We recall below the existing results on the global existence and decay results for the problem (1.1) with $\beta = 0$ and small initial datum.

Theorem 1.1 ([8]). *Consider the equation* (1.1) *with* $\beta = 0$ *supplemented with initial data*