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DISCONTINUOUS GALERKIN METHODS AND THEIR ADAPTIVITY FOR THE TEMPERED FRACTIONAL (CONVECTION) DIFFUSION EQUATIONS*

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Abstract

This paper focuses on the adaptive discontinuous Galerkin (DG) methods for the tempered fractional (convection) diffusion equations. The DG schemes with interior penalty for the diffusion term and numerical flux for the convection term are used to solve the equations, and the detailed stability and convergence analyses are provided. Based on the derived posteriori error estimates, the local error indicator is designed. The theoretical results and the effectiveness of the adaptive DG methods are, respectively, verified and displayed by the extensive numerical experiments. The strategy of designing adaptive schemes presented in this paper works for the general PDEs with fractional operators.

Mathematics subject classification: 26A33, 65M60, 65M12.
Key words: Adaptive DG methods, Tempered fractional equations, Posteriori error estimate.

1. Introduction

Fractional calculus [8] is a popular mathematical tool for modelling anomalous diffusions [27], being ubiquitous in nature. Microscopically, anomalous diffusion can be described by continuous time random walk (CTRW), defined by the waiting time and jump length; generally the first moment of the waiting time and/or the second moment of the jump length diverge(s). Sometimes, it is better to temper the broad distribution(s) of the waiting time and/or the jump length [4, 19, 25, 44], because of the boundedness of physical space or the finite lifespan of the biological particles or the slow transition of different diffusion types. Based on the tempered CTRW, the partial differential equations (PDEs) characterizing the evolution of the functional distribution of the trajectories of the particles are derived [41], which reduce to the PDEs describing the distribution of the positions of the particles if taking the parameter p over there as 0, called tempered fractional PDEs; here, we discuss their (adaptive) discontinuous Galerkin (DG) methods.

There are already some works for numerically solving (tempered) fractional PDEs by variational methods [19,22,26,29,31,33,40,42,46]. Ervin and Roop [22] firstly present the variational formulation for the fractional advection dispersion equation. The DG methods are particularly applied to fractional problems with their majority of characteristics [7,13,24,34,39,43], naturally being formulated for any order of accuracy in any element, being flexible in choosing element sizes in any place, suitable for adaptivity, being local and easy to invert for mass matrix, leading

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to an explicit formulation for time dependent problems, etc. Cockburn and Mustapha [14] provide a hybridizable DG method for fractional diffusion problems; McLean and Mustapha [29] discuss the superconvergence of the DG method for the fractional diffusion and wave equations; Xu and Hesthaven [42], and Wang et al [40], respectively, consider DG and hybridized DG methods for the fractional convection-diffusion equations; Zayernouri and Karniadakis [46] design discontinuous spectral element methods for the time and space fractional differential equations. Du et al [20] give a convergent adaptive finite element algorithm for nonlocal diffusion and peridynamic models. Ainsworth and Glusa [3], and Chen et al [11], discuss the adaptive algorithms about fractional Laplacian with integral definition and spectral definition, respectively. Zhao et al [45] design an adaptive algorithm for Riesz fractional derivative with a posteriori error estimator based on gradient recovery approach. It seems that there are not works for investigating the potential advantages of DG methods in adaptivity for fractional problems, by deriving posteriori error estimates and providing the local error indicators.

The model we consider in this paper is the two dimensional space tempered fractional differential equation with absorbing boundary conditions [17, 18, 21], i.e.,

$$\begin{cases} \partial_t u + \mathbf{b}\cdot\nabla u - \kappa_1 \nabla_x^{\alpha,\lambda} u - \kappa_2 \nabla_y^{\beta,\lambda} u = f, & (\mathbf{x},t) \in \Omega \times J, \\ u(\mathbf{x},t) = u_{\text{in}}, & \Gamma_{\text{in}} \times J, \\ u(\mathbf{x},0) = u_0(\mathbf{x}), & \mathbf{x} \in \Omega, \\ u(\mathbf{x},t) = 0, & (\mathbf{x},t) \in \mathbb{R}^2 \backslash \bar{\Omega} \times J, \end{cases} \tag{1.1}$$

where $\alpha, \beta \in (0,1)$, $\lambda > 0$, and $\kappa_1, \kappa_2 > 0$ in the domain $\Omega = [a,b] \times [c,d]$ and $J = [0,T]$. The boundary of the domain Ω is decomposed into two parts: the inflow part Γ_{in} and outflow part Γ_{out} defined by

$$\Gamma_{\text{in}} = \{\mathbf{x} \in \partial\Omega : \mathbf{b}\cdot\mathbf{n} < 0\}, \qquad \Gamma_{\text{out}} = \partial\Omega \backslash \Gamma_{\text{in}}, \tag{1.2}$$

where $\mathbf{n}$ is the unit outward normal vector on the boundary. The model (1.1) describes a convection-diffusion problem with convection term $\mathbf{b}\cdot\nabla u$ and diffusion term $\kappa_1 \nabla_x^{\alpha,\lambda} u + \kappa_2 \nabla_y^{\beta,\lambda} u$ in horizontal and vertical directions respectively [17, 28]. Because of the existence of nonlocal operators $\nabla_x^{\alpha,\lambda}$ and $\nabla_y^{\beta,\lambda}$, the local boundary $\partial\Omega$ itself cannot be hit by the majority of discontinuous sample trajectories; based on this physical implication, this problem should be specified the generalized Dirichlet boundary conditions, i.e., in the complementary of Ω [17, 18, 21]. Note that if $\kappa_1 = \kappa_2$ and $\alpha = \beta$, the diffusion term will not reduce to a two-dimensional fractional Laplacian [17]. The function $f \in L^2(J; L^2(\Omega))$ is a source term; the convection coefficient $\mathbf{b}$ is assumed to be continuous and satisfy $\nabla\cdot\mathbf{b} = 0$, and the initial function $u_0 \in L^2(\Omega)$. The tempered fractional operator $\nabla_x^{\alpha,\lambda}$ is defined from [9] and will be shown in the next section. Compared with non-tempered case, the tempered operator $\nabla_x^{\alpha,\lambda}$ characterizes the physical reality that the jump length of a particle will not be arbitrarily large [41]. As for the discussion of the adaptivity of the fractional problems, we start from the steady state version of (1.1) with $\mathbf{b} = \mathbf{0}$. The first part of the paper focuses on designing the DG scheme of (1.1) with genuinely triangular grids, and offering explicit theoretical analyses. Being different from [33], which constructs the LDG scheme by rewriting the fractional equation as a first order system, we adopt the primal DG methods, namely interior penalty (IP) method, still keeping the advantages over the classical continuous Galerkin method in facilitating hp-adaptivity and yielding block diagonal mass matrices in time-dependent problems. Generally, the non-ignorable drawback of the IP method is to specify sufficient large penalty parameter for guaranteeing numerical stability, which degrades the performance of the iterative solver of the linear system [36]. Fortunately, for the (tempered) fractional equations, this drawback disappears, since the schemes are stable