Journal of Computational Mathematics
Vol.38, No.6, 2020, 870–880.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1906-m2019-0051

THE PLATEAU-BÉZIER PROBLEM WITH WEAK-AREA FUNCTIONAL*

Yongxia Hao
Faculty of Science, Jiangsu University, Zhenjiang, China
Email: yongxiahaoujs@ujs.edu.cn

Abstract

In this paper, we present a new method to solve the Plateau-Bézier problem. A new energy functional called weak-area functional is proposed as the objective functional to obtain the approximate minimal Bézier surface from given boundaries. This functional is constructed based on Dirichlet energy and weak isothermal parameterization condition. Experimental comparisons of the weak-area functional method with existing Dirichlet, quasi-harmonic, the strain energy-minimizing, harmonic and biharmonic masks are performed which show that the weak-area functional method are among the best by choosing appropriate parameters.

Mathematics subject classification: 65D17, 65D18.
Key words: Minimal surface, Plateau-Bézier problem, Weak isothermal parameterization, Weak-area functional.

1. Introduction

The problem of finding a surface that minimizes the area with prescribed border is called the Plateau problem [3,4,15,16]. Such surfaces are called minimal surfaces and characterized by the fact that the mean curvature vanishes. The minimal surface has attracted scientists for many years and has been studied extensively in many literatures, such as [5,12,13,21–23]. Part of the interest stems from the fact that it is so easily realizable physically in the form of soap films, and for this reason it has been studied not only mathematically, but also physically for many years [19]. The fascinating characters of minimal surface make it to be widely used in many areas such as architecture, material science, ship manufacture, biology and so on [17]. For instance, architecture inspired from minimal surface embodies the unite of economy and beauty. Furthermore, scientists and engineers have anticipated the nanotechnology applications of minimal surface in areas of molecular engineering and materials science [20]. Applications of minimal surface in aesthetic design have also been presented in [18].

As we know, only a few minimal surfaces have been found in closed form. Hence, numerical methods have been devised to construct approximate minimal surface. Brakke proposed an approach to compute a parametric minimal surface with the finite element method [1]. Direct simulation of surface tension forces on a grid of marker particles is used for the minimal surface approximation in [2,9]. Jung et al. proposed a variational level set approach for the surface area minimization of triply-periodic surfaces [10]. Tråsdahl and Rønquist presented an algorithm for finding high order numerical approximations of minimal surfaces with a fixed boundary [19].

In order to find an approximate Bézier solution of the Plateau problem, J. Monterde proposed the Plateau-Bézier problem [12], which is to find the surface of minimal area from among

* Received February 26, 2019 / Accepted June 20, 2019 /
Published online September 18, 2019 /

all Bézier surfaces with given boundary curves. Because of the high nonlinearity of the area functional, several energy functionals are used to approximate the area functional, which lead to easy management for the Plateau-Bézier problem. The first one is the Dirichlet functional employed as a replacement to solve the Plateau-Bézier problem in [12]. Based on this functional, a multiresolution analysis method with B-splines is proposed to obtain the parametric surface of minimal area in [7]. Moreover, the minimal quasi-Bézier surfaces in non-polynomial space are also investigated by the Dirichlet method and harmonic method in [8]. A new energy functional called quasi-harmonic energy functional is proposed in [21] as the objective functional to obtain the quasi-harmonic Bézier surface from given boundaries. Bending energy functional [11] and mean curvature energy functional [24] are also used for approximating the solution of the Plateau-Bézier problem.

Harmonic surface is related to minimal surface. The corresponding Euler-Lagrange equation of the Dirichlet functional is $\Delta \mathbf{r} = 0$, which defines the harmonic surface. Therefore, harmonic Bézier surface and biharmonic Bézier surface are also proposed as an approximation solution of the Plateau-Bézier problem [14]. A surface with isothermal parameterization is minimal surface if and only if it is harmonic surface. This is exactly the theoretical basis of the Dirichlet functional and quasi-harmonic functional to replace the area functional. However, both these two functionals are constructed without any thought of the isothermal parameterization. Therefore in this paper, we introduce a new energy functional constructed based on Dirichlet functional and isothermal parameterization to solve the Plateau-Bézier problem.

The remainder of this paper is organized as follows. Some preliminaries and weak-area energy functional are introduced in Section 2. Section 3 presents the sufficient and necessary conditions for Bézier surfaces with minimal weak-area energy. Some comparisons among different methods are presented in Section 4. Finally, we conclude and list some future works in Section 5.

2. Preliminary and Weak-Area Functional

In this section, we shall review some concepts and results related to minimal surfaces [15,16], and introduce the weak isothermal parameterization and weak-area functional.

2.1. Preliminary

For a parametric surface $\mathbf{r}(u, v)$, the coefficients of the first fundamental form are

$$E = \langle \mathbf{r}_u, \mathbf{r}_u \rangle, \qquad F = \langle \mathbf{r}_u, \mathbf{r}_v \rangle, \qquad G = \langle \mathbf{r}_v, \mathbf{r}_v \rangle,$$

where $\mathbf{r}_u, \mathbf{r}_v$ are the first-order partial derivatives of $\mathbf{r}(u, v)$ with respect to u and v respectively, and $\langle , \rangle$ defines the dot product of the vectors. The coefficients of the second fundamental form of $\mathbf{r}(u, v)$ are

$$L = (\mathbf{r}_u, \mathbf{r}_v, \mathbf{r}_{uu}), \qquad M = (\mathbf{r}_u, \mathbf{r}_v, \mathbf{r}_{uv}), \qquad N = (\mathbf{r}_u, \mathbf{r}_v, \mathbf{r}_{vv}),$$

where $\mathbf{r}_{uu}, \mathbf{r}_{vv}$ and $\mathbf{r}_{uv}$ are the second-order partial derivatives of $\mathbf{r}(u, v)$ and $(, ,)$ defines the mixed product of the vectors. Then the mean curvature H and the Gaussian curvature K of $\mathbf{r}(u, v)$ are

$$H = \frac{EN - 2FM + LG}{2(EG - F^2)}, \qquad K = \frac{LN - M^2}{EG - F^2}.$$