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Numerical Method for the Deterministic Kardar-Parisi-Zhang Equation in Unbounded Domains

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Abstract. We propose an artificial boundary method for solving the deterministic Kardar-Parisi-Zhang equation in one-, two- and three dimensional unbounded domains. The exact artificial boundary conditions are obtained on the artificial boundaries. Then the original problems are reduced to equivalent problems in bounded domains. A finite difference method is applied to solve the reduced problems, and some numerical examples are provided to show the effectiveness of the method.

Key words: Quasilinear parabolic equation; artificial boundary condition; viscous Hamilton-Jacobi equation; unbounded domain.

1 Introduction

Surface growth is a class of important problems arising from many practical applications [22], such as molecular beam epitaxy, bacterial growth, fluid flow in porous media or evolution of fire fronts, etc. During the recent two decades, many models based on stochastic partial differential equations have been developed to simulate the mechanism of surface growth. Among these, one of the most well known models is the one introduced by Kardar, Parisi and Zhang (KPZ) [18],

$$u_t = \nu\Delta u + \lambda|\nabla u|^2 + \eta, \tag{1.1}$$

where $u = u(x,t)$ represents the height of surface growth at $d-$dimensional position x

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and time t, ν and λ are the parameters of diffusion and nonlinear terms, respectively. The last term of the equation (1.1), $\eta = \eta(x,t)$, is a Gaussian white noise which is produced by a stochastic force.

The KPZ equation is the first continuum partial differential equation to model the dynamics of surface growth. Based on it numerous studies have been carried out by many authors. Among them a relevant problem is to study the dynamics of the KPZ equation without noise term, which describes the relaxation of an initially rough surface to a flat one.

In this paper, we study the initial boundary value problem of the deterministic KPZ equations with a source term in unbounded domains,

$$u_t = \Delta u + |\nabla u|^2 + f(x,t), \quad \text{in } \mathbb{R}^d \times (0,T] \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^d, \tag{1.3}$$

$$u \to 0, \quad \text{when } |x| \to +\infty, \tag{1.4}$$

where $d = 1, 2$, or 3, and the initial value $u_0(x)$ and the source term $f(x,t)$ vanish outside a d-dimensional ball $B_0^d = \{x : |x| \leq R\}$, namely,

$$\text{supp}\{f(x,t)\} \subset B_0^d \times [0,T], \quad \text{supp}\{u_0(x)\} \subset B_0^d. \tag{1.5}$$

For the numerical solution of problem (1.2)-(1.4), we need to introduce artificial boundaries to make the domain finite, to find the artificial boundary conditions, and to reduce the original problem to an equivalent problem on a bounded domain. The so-called artificial boundary method has been the most efficient method for the numerical solution of PDEs in an unbounded domain, including applications to wave equations [3, 8, 9, 19], elliptic equations [4,11,12,25] and, most relevant to the current work, the parabolic equations [7, 13,14,21,24], etc. In general, the basic assumption of the artificial boundary method is that the equation is linear. Consequently, certain analytic forms of the boundary conditions on the artificial boundaries can be obtained. Usually, the artificial boundary method cannot be applied directly to nonlinear problems. However, for some problems, if the equation can be linearized outside the artificial boundaries, then it is possible to find the boundary conditions on the artificial boundaries [5, 10, 15].

The artificial boundary method of the KPZ equation is an extension of the existing method for linear parabolic equations. Because the source term $f(x,t)$ is compact, the KPZ equation (1.2) can be transformed into a linear parabolic equation in the exterior domain where the artificial boundary condition can be derived. After transforming it back into the original variable, we can solve the problem on the finite domain. In the one-dimensional case, we have used this idea to solve the Burgers equation in unbounded domains [16].

Many numerical methods have been used for solving the stochastic KPZ equation (1.1) in bounded domains, such as finite difference methods and their improved versions [1, 20], pseudospectral methods [6] etc. For the deterministic equation, an effective method is the second order Crank-Nicolson scheme. The stability and convergence for the artificial