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A Space-Time Conservative Method for Hyperbolic Systems with Stiff and Non Stiff Source Terms[†]

Shamsul Qamar[1,*] and Gerald Warnecke[1]

[1] *Institute for Analysis and Numerics, Otto-von-Guericke University, PSF 4120, D-39106 Magdeburg, Germany.*

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Abstract. In this article we propose a higher-order space-time conservative method for hyperbolic systems with stiff and non stiff source terms as well as relaxation systems. We call the scheme a slope propagation (SP) method. It is an extension of our scheme derived for homogeneous hyperbolic systems [1]. In the present inhomogeneous systems the relaxation time may vary from order of one to a very small value. These small values make the relaxation term stronger and highly stiff. In such situations underresolved numerical schemes may produce spurious numerical results. However, our present scheme has the capability to correctly capture the behavior of the physical phenomena with high order accuracy even if the initial layer and the small relaxation time are not numerically resolved. The scheme treats the space and time in a unified manner. The flow variables and their slopes are the basic unknowns in the scheme. The source term is treated by its volumetric integration over the space-time control volume and is a direct part of the overall space-time flux balance. We use two approaches for the slope calculations of the flow variables, the first one results directly from the flux balance over the control volumes, while in the second one we use a finite difference approach. The main features of the scheme are its simplicity, its Jacobian-free and Riemann solver-free recipe, as well as its efficiency and high of order accuracy. In particular we show that the scheme has a discrete analog of the continuous asymptotic limit. We have implemented our scheme for various test models available in the literature such as the Broadwell model, the extended thermodynamics equations, the shallow water equations, traffic flow and the Euler equations with heat transfer. The numerical results validate the accuracy, versatility and robustness of the present scheme.

Key words: Hyperbolic systems with relaxation; stiff systems; space-time conservative and Jacobian-free method; high order accuracy; discontinuous solutions.

*Correspondence to: Shamsul Qamar, Institute for Analysis and Numerics, Otto-von-Guericke University, PSF 4120, D-39106 Magdeburg, Germany. Email: `Shamsul.Qamar@Mathematik.Uni-Magdeburg.DE`
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1 Introduction

Hyperbolic conservation laws with source terms arise in a variety of manners. They describe non-equilibrium flows with relaxation models, reacting flows and flows with phase transition. There are a variety of physical phenomena where hyperbolic systems with relaxation arise namely, nonlinear waves [41, 46], gas flows with relaxation [16], viscoelasticity [38], multiphase and phase transitions [40], reacting Euler equations [21], discrete velocity models in kinetic theory [18], gas with vibrational degrees of freedom [45], hyperbolic models for semiconductors [2, 4], reactive flows and radiation hydrodynamics [29]. The development of efficient numerical schemes for these hyperbolic systems is quite challenging, since in many applications the relaxation time of the source term varies from 1 to very small values if one compares it with the characteristic time scale of the hyperbolic system.

The study of relaxation problems was initiated by Whitham [46] for linear problems. For nonlinear hyperbolic systems of two equations the stability of the equilibrium equation, under the subcharacteristic condition, and the zero relaxation limit were proved by Liu [28] and Chen, Levermore and Liu [13], respectively.

It is usually impossible to separate physical problems into non-stiff and stiff regimes, especially for multiple relaxation times, hence one needs a solution to the full relaxation system in all possible cases. The construction of the schemes that work for all ranges of the relaxation time, using coarse grids that do not resolve the small relaxation time, has been studied mainly in the context of upwind methods using a method of line approach combined with suitable splitting techniques [9, 23]. Alternatively, the approaches based on the method of characteristics were also considered [6, 34]. Most of these schemes are based on the solution to the Riemann problem and, except for the scheme proposed in [9], do not provide uniform second-order accuracy with respect to the relaxation parameter without resolving the small scales or initial layer. Numerical schemes for hyperbolic systems with more general stiff relaxation terms have also been studied in [21, 24, 31].

More recently Liotta, Romano and Russo [27] extended the central schemes [32] to hyperbolic systems with source terms. They derived and analyzed both the explicit and implicit formulations. Later on, Pareschi [34] extended the schemes in order to treat the stiff sources in more efficient manner. Apart from central schemes, Pareschi and Russo [35] developed implicit-explicit (IMEX) Runge-Kutta methods up to order 3 for hyperbolic systems with source terms. In both central and Runge-Kutta schemes they obtained excellent results especially for the stiff source terms where the behavior of the scheme is more important to be analyzed. We also consider the same numerical test cases with the same parameters which were presented in these articles. Hence one may compare our results with those presented in [23, 27, 34].

Apart from the above mentioned schemes there is another family of schemes called space-time conservation element and solution element (CE/SE) methods of Chang [10]. Like central schemes, these schemes also do not need Riemann solvers. However, unlike the upwind schemes and central schemes, the flow variable distribution inside the solution