An r-Adaptive Finite Element Method for the Solution of the Two-Dimensional Phase-Field Equations

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Abstract. An adaptive moving mesh method is developed for the numerical solution of two-dimensional phase change problems modelled by the phase-field equations. The numerical algorithm is relatively simple and is shown to be more efficient than fixed grid methods. The phase-field equations are discretised by a Galerkin finite element method. An adaptivity criterion is used that ensures that the mesh spacing at the phase front scales with the diffuse interface thickness.

Key words: Phase change; phase-field; equidistribution; moving meshes; adaptive method.

1 Introduction

There has been much recent interest in the modelling of solidification processes. The main challenge is to incorporate events on the smallest microstructural scales to the larger macroscopic scales. Classical Stefan models do not take account of important physical properties such as undercooling and surface tension. These effects are normally incorporated within modified Stefan models (see Section 2 below). The numerical simulation of the modified model requires the estimation of the curvature of the interface between the solid and liquid phases. This is often a difficult task, especially in three dimensions or when phase fronts merge.

Modified Stefan problems have become more amenable to numerical solution by the introduction of phase-field models. Front-tracking is avoided by introducing an auxiliary continuous order parameter p that interpolates between the solid and liquid phases, attaining two different constant values in each phase (e.g, $p = \pm 1$), with a rapid transition

region at the solidification front. The level set $p = 0$ is identified with the front and p is assumed to evolve in such a way that it minimises a free energy functional consistent with thermodynamics.

Most numerical methods used to solve the phase-field equations have utilised stationary uniform meshes [9,20,29,31]. However, it is well known that it is important that the diffuse interface is well resolved if the correct dynamics are to be reproduced. As the phase interface moves in time it is clear that an efficient numerical method must use some form of mesh adaptivity. Within a finite element context this is usually achieved using the h-method of adaptation, where the mesh is locally refined or coarsened by adding or deleting points. This strategy has been used successfully in [6, 25, 26]. A less popular approach is to use the so-called r-adaptive method where mesh points are moved throughout the domain while the connectivity of the mesh is kept fixed. The main reason for the lack of popularity of this approach is the difficulty involved in controlling the geometric quality of the mesh elements. However, the development of a robust r-adaptive method is attractive in that it intuitively should be able to accurately resolve and follow important solution features. The coding involved in an r-adaptive method is also simpler than an h-method, which requires a considerably more complicated data structure.

In this paper we will concentrate on an r-adaptive method where movement of the mesh is based on a variational formulation used by Huang and Russell [18], Cao *et al.* [10], and Huang [17]. The basic idea is to move the mesh so that it attempts to minimise a weighted quadratic functional where the weights are based on some local adaptivity criterion. Such an approach has been successfully used to solve a regularised formulation of two-dimensional classical Stefan problems [4] and convective heat transfer problems involving a change of phase [28]. Moving mesh methods have also been used to solve the phase-field equations in one dimension [22, 23] and in two dimensions [27]. In particular, in [22] we have been able to suggest an appropriate adaptivity criterion based on an asymptotic expansion of the interface region of a planar travelling wave solution. The resulting mesh has been shown to automatically scale with the diffuse interface thickness.

The main aim of this paper is to investigate the use of a moving mesh approach to solve the two-dimensional phase-field equations. The layout of this paper is as follows: in the next section we present the sharp and diffuse interface models for heat conduction with a change of phase. In Section 3 we discuss how the moving mesh is generated along with specific adaptivity criteria for the phase-field equations. In Section 4 we describe a semi-implicit moving finite element discretisation of the phase-field model. Finally, we apply the moving mesh method to a number of test cases in Section 5.

2 Sharp and diffuse interface models

Let $\Omega \in \mathbb{R}^2$ be a bounded domain with a Lipschitz continuous boundary $\partial\Omega$. For each $t \in \mathbb{R}^+$ we will assume we have a decomposition of Ω into subdomains $\Omega^+(t)$ and $\Omega^-(t)$ so that $\Omega = \Omega^+(t) \cup \Omega^-(t) \cup \Gamma(t)$, where the interface $\Gamma(t) = \partial\Omega^+(t) \cup \partial\Omega^-(t)$ is smooth.