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Elements of the Lattice Boltzmann Method II: Kinetics and Hydrodynamics in One Dimension

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Abstract. Concepts of the lattice Boltzmann method are discussed in detail for the one-dimensional kinetic model. Various techniques of constructing lattice Boltzmann models are discussed, and novel collision integrals are derived. Geometry of the kinetic space and the role of the thermodynamic projector is elucidated.

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1 Introduction

In the first paper of this series [1], we have discussed some primary concepts of the lattice Boltzmann method for solving partial differential equations. The goal of the present paper is to extend the introduction of the lattice Boltzmann method to nonlinear problems while keeping the presentation as elementary as possible.

The outline of the paper is as follows. In Section 2 we consider the one-dimensional Navier-Stokes equations, and identify the requirements for lifting them to a kinetic equation. Construction of the kinetic equation begins in Section 3 where we derive the pertinent entropy function. In Section 4, we derive the corresponding equilibrium. In Section 5, we describe geometry of the phase space of kinetic equations, hydrodynamic and kinetic subspaces, and introduce the notion of detail balance as a geometrical statement. This section contains preliminary information which is used in the construction of collision integrals (Section 6). We develop general methods of constructing admissible collision integrals based on the entropy function. In Section 7, we consider linearization of

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collision integrals at equilibrium, and discuss in detail the notion of thermodynamic projector. In Section 8 we consider a special class of collision integrals which have the feature that their linearization is spectrally equivalent to the linearized Bhatnagar-Gross-Krook kinetic model (single relaxation time gradient models). In Section 9 we consider the entropic lattice Boltzmann scheme for these new models, and give a thorough analysis of the hydrodynamic limit of the discrete-time kinetic equation. We conclude in Section 10 with a brief discussion.

Finally, we did every effort to make the presentation self-containing, thus, references are kept at a minimal level. For a further reading on the lattice Boltzmann method, we direct the reader to the papers [2–5] and reviews [6–9]. Development of the entropic lattice Boltzmann method can be found in [10–20].

2 Hydrodynamic and kinetic equations

2.1 Navier-Stokes equations in one dimension

The target equations are the balance equations for the density $\rho(x,t)$ and the momentum density $j(x,t)=\rho u(x,t)$:

$$\partial_t\rho+\partial_x(\rho u)=0, \tag{2.1}$$

$$\partial_t(\rho u)+\partial_x P=0, \tag{2.2}$$

$$P=\rho c_{\mathrm{s}}^2+\rho u^2-2\nu\rho\partial_x u. \tag{2.3}$$

This is the simplest example of the *Navier-Stokes equations*. We have written them in the 'conservation laws + constitutive equation' form. Now we have two equations for the conservation laws (for the density ρ and for the momentum j). The constitutive equation for the pressure P (2.3) consists of two parts. The first part, P^{E},

$$P^{\mathrm{E}}=\rho c_{\mathrm{s}}^2+\rho u^2, \tag{2.4}$$

is the value of the pressure at the equilibrium. If (2.4) is substituted instead of P in the balance equation for the momentum (2.2), the resulting non-dissipative hydrodynamic equations (2.1) and (2.2) form the simplest set of *Euler equations*. The second part of the pressure, P^{neq} is the non-equilibrium contribution,

$$P^{\mathrm{NS}}=-2\nu\rho\partial_x u. \tag{2.5}$$

Parameter $\nu>0$ is the *viscosity coefficient*. While in the Eqs. (2.3) and (2.5) the viscosity coefficient appears simply as a parameter, we can infer that it will be expressed in terms of kinetic parameters of the kinetic models (relaxation time) once we will write it down (the same happened to the diffusion coefficient in the example of the advection-diffusion equation in [1]). The form of the constitutive relation (2.5) where the non-equilibrium pressure is proportional to the gradient of the momentum is typical of the *Newtonian fluid*.