REVIEW ARTICLE

Multi-Valued Solution and Level Set Methods in Computational High Frequency Wave Propagation

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Abstract. We review the level set methods for computing multi-valued solutions to a class of nonlinear first order partial differential equations, including Hamilton-Jacobi equations, quasi-linear hyperbolic equations, and conservative transport equations with multi-valued transport speeds. The multivalued solutions are embedded as the zeros of a set of scalar functions that solve the initial value problems of a time dependent partial differential equation in an augmented space. We discuss the essential ideas behind the techniques, the coupling of these techniques to the projection of the interaction of zero level sets and a collection of applications including the computation of the semiclassical limit for Schrödinger equations and the high frequency geometrical optics limits of linear wave equations.

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1 Introduction

In the computation of wave propagation, when the wave field is highly oscillatory, direct numerical simulation of the wave dynamics can be prohibitively costly and approximate models for wave propagation must be used. The resulting approximate models are often nonlinear, and the corresponding classical entropy or viscosity type solutions are not adequate in describing the wave behavior beyond the singularity, where multi-valued solutions in physical space are needed. Therefore, capturing multi-valued solutions by efficient algorithms is an important issue. Examples include dispersive waves [30, 40, 62], optical waves [17, 18], seismic waves [23, 57, 61], semiclassical limits of Schrödinger equations [10, 33, 56], electron beam modulation in vacuum electronic devices [41], etc. More applications arise constantly.

The level set method has been a highly successful computational technique for capturing the evolution of curves and surfaces [49, 50] with applications in diverse areas such as multi-phase fluids, computer vision, imaging processing, optimal shape design, etc. This paper reviews a newly developed level set framework for the computation of multivalued solutions of a large class of nonlinear PDEs that are encountered in various high frequency wave propagation problems mentioned above. We hope that this article will help the community understand how ideas of the level set method have been used in this challenging area, and the problems that remain in extending this method to other physical applications.

1.1 Asymptotic methods

We consider a complex wave field $u^\epsilon(x, t)$ governed by a linear wave type equation, say the Schrödinger equation

$$i\epsilon u_t^\epsilon = -\frac{\epsilon^2}{2}\Delta u^\epsilon + V(x)u^\epsilon,$$

where $V(x)$ is a given potential, and $\epsilon > 0$ denotes a re-scaled Planck constant. Here the regime of interest is the so called *semiclassical approximation* where ϵ tends to zero. A widely used classical approach is the so called WKB method or geometrical optics, which uses asymptotic approximations obtained when the small scale goes to zero.

The derivation of the WKB system comes through a formal expression

$$u^\epsilon(x, t) = A^\epsilon(x, t)\exp(iS(x, t)/\epsilon). \tag{1.1}$$

Assuming that the phase S and the amplitude A^ϵ are sufficiently smooth, we expand the amplitude in powers of ϵ:

$$A^\epsilon = A_0 + \epsilon A_1 + \epsilon^2 A_2 + \cdots.$$

Insertion of this expression into the underlying linear wave equation and balancing terms of $\mathcal{O}(1)$ order in ϵ gives separate equations for A and S.