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Explicit Multi-Symplectic Methods for Hamiltonian Wave Equations

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Abstract. In this paper, based on the multi-symplecticity of concatenating symplectic Runge-Kutta-Nyström (SRKN) methods and symplectic Runge-Kutta-type methods for numerically solving Hamiltonian PDEs, explicit multi-symplectic schemes are constructed and investigated, where the nonlinear wave equation is taken as a model problem. Numerical comparisons are made to illustrate the effectiveness of our newly derived explicit multi-symplectic integrators.

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Key words: Hamiltonian wave equations, multi-symplectic integration, symplectic Runge-Kutta methods, symplectic Runge-Kutta-Nyström methods.

1 Introduction

Consider the following Hamiltonian partial differential equation (PDE),

$$K\partial_t z + L\partial_x z = \nabla_z S(z), \tag{1.1}$$

where $z \in \mathbb{R}^n$, $S: \mathbb{R}^n \mapsto \mathbb{R}$ is some smooth function and K, L are two skew-symmetric constant $n \times n$ matrices. It is well-known that system (1.1) is multi-symplectic, i.e., its phase flow gives rise to the multi-symplectic conservation law

$$\partial_t \omega + \partial_x \kappa = 0, \tag{1.2}$$

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with

$$\omega = dz \wedge Kdz, \qquad \kappa = dz \wedge Ldz.$$

The local invariant (1.2) implies that symplecticity can vary over the spatial domain from time to time. But this variation is not arbitrary as the changes in time are exactly compensated by changes in space (see [9, 12], and references therein). Some very important PDEs can be rewritten in this form, particularly, including various wave equations (see, e.g., [2, 8, 9, 14] and references therein).

More recently, there has been growing interest in multi-symplectic integration for Hamiltonian PDEs (1.1), i.e., numerical integrators whose numerical flow gives rise to certain preservation of the local invariant (1.2). Now, for our study, we introduce a uniform grid $\{x_k, t_l\} \in \mathbb{R}^2$ with mesh length h in the x-direction and mesh length τ in the t-direction, which will be used throughout the paper. According to the definition in [2], a numerical discretization for (1.1), i.e.,

$$K\partial_t^{k,l} z_k^l + L\partial_x^{k,l} z_k^l = (\nabla_z S(z_k^l))_k^l, \qquad z_k^l \approx z(x_k, t_l), \tag{1.3}$$

where $\partial_t^{k,l}$ and $\partial_x^{k,l}$ are discretizations of the derivatives ∂_t and ∂_x, respectively, is called multi-symplectic, if it satisfies a discrete version of the multi-symplectic conservation law:

$$\partial_t^{k,l} \omega_k^l + \partial_x^{k,l} \kappa_k^l = 0, \tag{1.4}$$

where

$$\omega_k^l = dz_k^l \wedge Kdz_k^l, \qquad \kappa_k^l = dz_k^l \wedge Ldz_k^l,$$

and dz_k^l satisfies the discrete variational equation

$$K\partial_t^{k,l} dz_k^l + L\partial_x^{k,l} dz_k^l = S''(z_k^l) dz_k^l. \tag{1.5}$$

Some progress has been made on multi-symplectic integration for various Hamiltonian PDEs (see, e.g., [1,5,6,9,11,14] and references therein). In particular, as one of the most important classes of multi-symplecitic integrators, the concatenation of symplectic Runge-Kutta (SRK) methods and symplectic partitioned Runge-Kutta (SPRK) methods is intensively studied in [5, 6, 11, 14]. However, since both SRK methods and SPRK methods are implicit (this means that in the numerical implementation some iteration methods must be utilized for the nonlinear case) with only a very few exceptional cases of lower order methods [15], multi-symplectic integrators constructed in this way are usually fully implicit (see, e.g., [11]). Clearly, this brings numerous difficulties for practical implementations due to the massive computational costs and for this reason, we are led to the challenging problem of systematically constructing efficient explicit multi-symplectic integrators.

As mentioned before, both SRK and SPRK are implicit, and concatenations of implicit methods inevitably produce implicit methods for numerically solving PDEs. Therefore,