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Efficient Algorithms for Approximating Particular Solutions of Elliptic Equations Using Chebyshev Polynomials

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Abstract. In this paper, we propose efficient algorithms for approximating particular solutions of second and fourth order elliptic equations. The approximation of the particular solution by a truncated series of Chebyshev polynomials and the satisfaction of the differential equation lead to upper triangular block systems, each block being an upper triangular system. These systems can be solved efficiently by standard techniques. Several numerical examples are presented for each case.

AMS subject classifications: 33A65, 65N35, 65N22, 35J05

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1 Introduction

Boundary methods such as the Boundary Integral Equation Method (BIEM) [2, 5] and the Method of Fundamental Solutions (MFS) [12,16] are numerical techniques applicable for the numerical solution certain elliptic boundary value problems. In these methods, the dimension of the problem is reduced by one as only the boundary of the domain of the problem under consideration needs to be discretized. The advantages of these techniques can be fully exploited if the governing differential equation is homogeneous. It is therefore often desirable to convert an elliptic boundary value problem governed by an inhomogeneous differential equation to one governed by a homogeneous differential equation. This can be achieved using the Method of Particular Solutions (MPS). To

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describe the MPS, consider the boundary value problem

$$Lu=f \quad \text{in} \quad \Omega, \quad u=g \quad \text{on} \quad \partial\Omega, \tag{1.1}$$

where L is a second order linear elliptic operator and Ω is an open bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. If u_p is a particular solution of the governing equation, then it satisfies $Lu_p=f$ but does not necessarily satisfy the boundary condition. If we let $v=u-u_p$, then v satisfies the boundary value problem

$$Lv=0 \quad \text{in} \quad \Omega, \quad v=g-u_p \quad \text{on} \quad \partial\Omega. \tag{1.2}$$

Clearly, the governing equation is now homogeneous and thus problem (1.2) can be easily solved using a boundary-type method. In order to transform problem (1.1) into problem (1.2), we need to construct an approximation to the particular solution u_p.

In recent years, many methods have been proposed for the approximation of particular solutions. These methods may be classified as direct or indirect [10]. Direct methods approximate a solution of $Lu_p=f$ by a numerical method. For example, it is well-known that a particular solution of the Poisson equation $\Delta u_p=f$ in $\mathbb{R}^2$ is given by the Newtonian potential [1]

$$u_p(P)=\frac{1}{2\pi}\int_{\Omega}\log|P-Q|f(Q)dV(Q), \tag{1.3}$$

where $|P-Q|$ denotes the distance between the points P and Q. In general, the integral (1.3) cannot be evaluated analytically and so numerical integration is used. Since Ω can have an arbitrary shape, the numerical evaluation of the integral (1.3) requires a complicated domain discretization of Ω. To avoid the difficulties associated with such a discretization, Atkinson's method [1] may be used. In it, one assumes that f can be extended smoothly to $\tilde{\Omega}$, where $\Omega\subseteq\tilde{\Omega}$. Then $u_p(P)=\frac{1}{2\pi}\int_{\tilde{\Omega}}\log|P-Q|f(Q)dV(Q)$ is also a particular solution of $\Delta u_p=f$. The advantage of using this expression instead of (1.3) is that the domain $\tilde{\Omega}$ may be chosen so that the calculation of the integral is simplified [14]. The indirect approach for solving, for example, Poisson problems, is based on the Dual Reciprocity Method (DRM) [9, 17, 25]. In the DRM, the source term f is approximated by $\hat{f}=\sum_{i=1}^{n}a_i\hat{f}_i$, where $\{\hat{f}_i\}_{i=1}^{n}$ is an appropriate set of functions. An approximation to the particular solution u_p is obtained by taking $\hat{u}_p=\sum_{i=1}^{n}a_i\hat{u}_i$, where each $\hat{u}_i$ satisfies $\Delta\hat{u}_i=\hat{f}_i$. An appropriate set of functions is the set of Radial Basis Functions (RBFs) [7, 9, 15, 17, 18, 22]. The most popular RBFs are thin plate and higher order radial splines, multiquadrics and Gaussians which are all globally supported [9, 10, 17–19]. The problem is that these globally supported basis functions lead to dense systems which can be highly ill-conditioned [9]. This difficulty can be overcome by using compactly supported RBFs (CS-RBFs) which have been extensively discussed in [9, 17]. The most popular CS-RBFs are Wendland's CS-RBFs [9, 17]. Polynomials and trigonometric functions have also been used as basis functions [22]. With these sets of basis functions a number of numerical methods can be used for determining approximation $\hat{f}$ [9, 10, 17, 22]. The properties of orthogonal polynomials, such as Chebyshev and Legendre polynomials are