

## Modeling of the Frozen Mode Phenomenon and its Sensitivity Using Discontinuous Galerkin Methods

S. Chun and J. S. Hesthaven\*

*Division of Applied Mathematics, Brown University, Providence, RI 02912, USA.*

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**Abstract.** We investigate the behavior and sensitivity of the frozen mode phenomenon in finite structures with anisotropic materials, including both magnetic materials and non-normal incidence. The studies are done by using a high-order accurate discontinuous Galerkin method for solving Maxwells equations in the time domain. We confirm the existence of the phenomenon also in the time-domain and study carefully the impact of the finite crystal on the frozen mode. This sets the stage for a thorough study of the robustness of the frozen mode phenomenon, resulting in guidelines for which design parameters are most sensitive and acceptable tolerances.

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**Key words:** Discontinuous Galerkin method, Maxwell equation, photonic crystals, frozen mode.

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### 1 Introduction

The use of complex metamaterials for controlling the propagation and manipulation of electromagnetic energy continues to attract significant interest among engineers. Recently, there has been a flurry of activity in the study and development of periodic structures comprising of several different anisotropic materials after it was shown that such structures could support highly unusual electromagnetic phenomena [18–20] such as a non-reciprocal propagation, very low transmission loss into the crystal and perhaps a most interesting phenomenon known as the frozen mode.

The frozen mode is a distinctive phenomenon related to stationary inflection points of the dispersion relations  $\omega(k)$  such as,

$$\frac{\partial\omega}{\partial k}=0; \quad \frac{\partial^2\omega}{\partial k^2}=0; \quad \frac{\partial^3\omega}{\partial k^3}\neq 0 \quad \text{at } \omega=\omega_0.$$

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\*Corresponding author. *Email addresses:* schun@dam.brown.edu (S. Chun), Jan.Hesthaven@Brown.edu (J. S. Hesthaven)

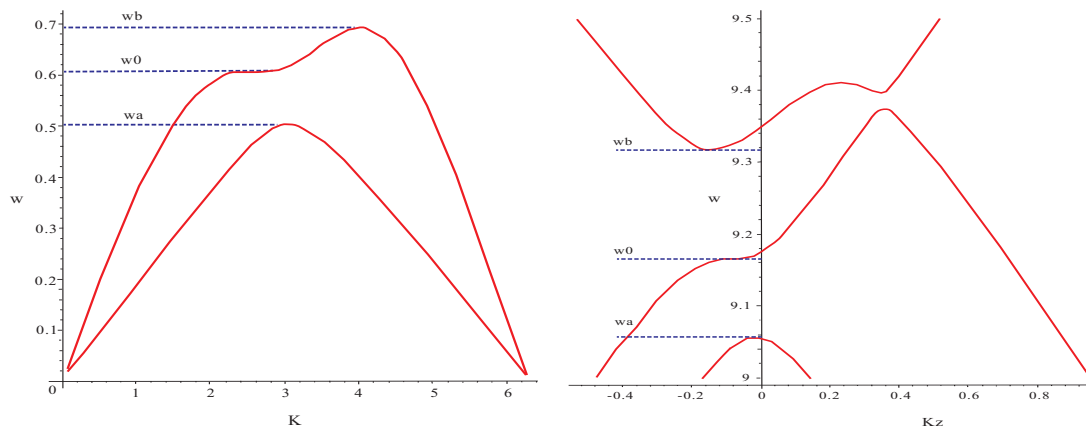


Figure 1: Dispersion relations for two different types of layer. Left: gyotropic photonic crystals ( $\omega_b = 0.690320778$ ,  $\omega_0 = 0.607676756$ ,  $\omega_a = 0.502044368$ ) and right: oblique anisotropic layers ( $\omega_b = 9.3160209805$ ,  $\omega_0 = 9.164450223$ ,  $\omega_a = 9.0569705995$ ).  $\omega$  and  $k$  are expressed in units of  $c/L$  and  $1/L$ .

When a monochromatic wave with frequency  $\omega_0$  propagates into a periodic array of unit cells with this dispersion relation, the wave shows the following striking features due to the significantly large transmittance rate at inflection points contrary to the negligible transmittance rate at band edges [18],

- dramatic slow-down of the waves
- enormously increased field amplitude
- cup-like singularity of the transmittance rate
- unidirectionality of monochromatic waves.

Let us briefly explain these features in the following. Fig. 1 shows two different types of dispersion relation for two different layers, but both of them contain the same kind of inflection point  $\omega_0$ . At each  $\omega_0$  in Fig. 1, we have two eigenmodes  $k_0$  and  $k_1$  such that  $\frac{\partial \omega}{\partial k}|_{k=k_0} = 0$ ,  $\frac{\partial \omega}{\partial k}|_{k=k_1} < 0$  and only the eigenmode at  $k_1$  transfers the energy. Since we have no eigenmode with a positive group velocity, the transmitted wave does not transfer the energy in the direction of the propagation. Physically, it means that the incident electromagnetic wave slows down with infinitesimal group velocity within the periodic layers. However, if a wave propagates in the opposite direction, we have  $\frac{\partial \omega}{\partial k}|_{k=k_1} > 0$  and the eigenmode  $k_1$  transfers the energy and consequently the abnormal slow-down vanishes. Thus, the crystal behaves differently depending on the vector of propagation, a phenomenon known as *electromagnetic unidirectionality*.

When the frequency of the wave is close to  $\omega_0$ , a transmitted wave consists of propagation components and evanescent components. The latter decays as the wave proceeds along the propagation direction, but the former remains to propagate. Both components initially increase dramatically, but their magnitudes remain almost equal with opposite sign. Thus, the field amplitude at the interface of the layer and vacuum remain almost the same, but once the wave precedes into the slab, the evanescent components die out