COMMUNICATIONS IN COMPUTATIONAL PHYSICS
Vol. 1, No. 5, pp. 847-858

Commun. Comput. Phys.
October 2006

# An Efficient Operator-Splitting Method for Noise Removal in Images

D. Krishnan[1,*], P. Lin[1] and X.-C. Tai[2]

[1] *Department of Mathematics, National University of Singapore, Singapore 117453.*
[2] *Department of Mathematics, University of Bergen, Johannes Brunsgate 12, Bergen, N-5008, Norway.*

Received 5 October 2005; Accepted (in revised version) 5 March 2006

**Abstract.** In this work, noise removal in digital images is investigated. The importance of this problem lies in the fact that removal of noise is a necessary pre-processing step for other image processing tasks such as edge detection, image segmentation, image compression, classification problems, image registration etc. A number of different approaches have been proposed in the literature. In this work, a non-linear PDE-based algorithm is developed based on the ideas proposed by Lysaker, Osher and Tai [*IEEE Trans. Image Process.*, 13 (2004), 1345-1357] . This algorithm consists of two steps: flow field smoothing of the normal vectors, followed by image reconstruction. We propose a finite-difference based additive operator-splitting method that allows for much larger time-steps. This results in an efficient method for noise-removal that is shown to have good visual results. The energy is studied as an objective measure of the algorithm performance.

**Key words**: Noise removal; nonlinear PDEs; additive operator splitting (AOS).

## 1 Introduction

In this work, removal of additive, zero-mean noise in digital images is investigated. We use the ideas proposed in [1], based on a TV-norm approach. This results in two nonlinear partial differential equations. The first of these equations is the smoothing of the flow-field (normal field) of the original image. The second equation reconstructs a noise-reduced image from the smoothed flow-field. This results in an efficient method for noise-removal that has good results.

*Correspondence to: D. Krishnan, Department of Mathematics, National University of Singapore, Singapore 117453. Email: dilipk@nus.edu.sg

 

The contributions of this paper are as follows: a more efficient scheme for flow-field smoothing is developed; an additive operator splitting (AOS) method ( [2, 15, 16]) is employed to further improve the efficiency of the flow-field smoothing; a comparison of the AOS method and explicit methods are done in terms of numerical performance; and lastly, the use of the energy is suggested as an objective measure of the performance of noise-reduction algorithms based on energy minimization.

Let $d$ be a digital image defined on a two-dimensional region $\Omega$. Let $(x, y)$ denote the position of a single pixel. $d(x, y)$ is the grey-level value associated with the pixel $(x, y)$. The noise model is assumed to be zero-mean and additive, denoted as $\eta(x, y)$. The observed image values $d_0(x, y)$ are

$$d_0(x, y) = d(x, y) + \eta(x, y). \tag{1.1}$$

The problem is to recover the (unknown) true image $d(x, y)$ from the given observations $d_0(x, y)$.

The visually annoying parts of the noise usually belong in the higher frequency regions of the spectrum. A lot of noise can be effectively filtered by a lowpass filter. However, this will remove the true high-frequency components of the image such as edges and texture. Isotropic filters suffer from this problem. The challenge is to retain as much of the true high-frequency information as possible while reducing the perceived noise levels in the image.

The Total Variation (TV) norm based filters proposed in [5], have been shown to be quite effective in removing noise without causing excessive smoothing of the edges. The original formulation of this filter is to obtain $d(x, y)$ as a solution of the constrained optimization problem

$$\inf_d \int_\Omega |\nabla d| \, dx \quad \text{subject to} \quad \int_\Omega |d - d_0|^2 \, dx = \sigma^2, \tag{1.2}$$

where $\sigma^2$ represents the noise level. The resulting Euler-Lagrange PDE to be solved in this case is

$$-\nabla \cdot \left( \frac{\nabla d}{|\nabla d|} \right) + \mu(d - d_0) = 0. \tag{1.3}$$

However, it is well known that the TV norm filter has the disadvantage of a *stair-case effect*: smooth functions get transformed into piecewise constant functions. This lends an undesirable blocky effect to the smoothed image. In [1], it is proposed to modify the equation (1.2). Instead of minimizing the TV norm of $d$, it is proposed in [1] to minimize the TV norm of $\nabla d/|\nabla d|$, giving the following equation

$$\inf_d \int_\Omega \left| \nabla \frac{\nabla d}{|\nabla d|} \right| \, dx \quad \text{subject to} \quad \int_\Omega |d - d_0|^2 \, dx = \sigma^2, \tag{1.4}$$

$\vec{n} = \nabla d/|\nabla d|$ is called the normal field of the image. The fourth-order Euler-Lagrange equation that results from directly minimizing this functional is difficult to solve numerically in a stable manner. Therefore, in [1], a re-formulation of this equation is done. The