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Simulation of Three-Dimensional Bénard-Marangoni Flows Including Deformed Surfaces

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Abstract. We present a coupled thermal-fluid model for Bénard-Marangoni convection in a three-dimensional fluid layer. The governing equations are derived in detail for two reasons: first, we do not assume a flat free surface as commonly done; and second, we prepare for the use of flexible discretizations. The governing equations are discretized using spectral elements in space and an operator splitting approach in time. Since we are here primarily interested in steady state solutions, the focus is on the spatial discretization. The overall computational approach is very attractive to use for several reasons: (i) the solution can be expected to have a high degree of regularity, and rapid convergence can be expected; (ii) the spectral element decomposition automatically gives a convenient parameterization of the free surface that allows powerful results from differential geometry to easily be exploited; (iii) free surface deformation can readily be included; (iv) both normal and tangential stresses are conveniently accounted for through a single surface integral; (v) no differentiation of the surface tension is necessary in order to include thermocapillary effects (due to integration-by-parts twice); (vi) the geometry representation of the free surface need only be C^0 across element boundaries even though curvature effects are included. Three-dimensional simulation results are presented, including the free surface deflection due to buoyancy and thermocapillary effects.

AMS subject classifications: 65C20, 76D05, 76D45, 76E06, 65N35, 53A05

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1 Introduction

We consider here Bénard-Marangoni convection where a fluid layer is heated from below. This problem has previously been studied extensively experimentally, theoretically, and

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computationally. One of the intriguing features with this problem is the formation of hexagonal convection cells from random initial conditions; see [2,14,25]. This formation can originate from different effects: it can be caused by small density variations due to the fact that the density is a function of the temperature (i.e., from buoyancy forces), or it can be due to variations in the surface tension due to the fact that the surface tension is a function of the temperature (i.e., thermocapillary forces), or both effects can be present at the same time.

However, the free surface deformation associated with these cells has previously only been studied experimentally [2,6,14] or analyzed analytically using linear stability analysis [23]. It is known experimentally that the free surface will either be depressed or elevated over each hexagonal cell, depending on whether surface-tension-gradient effects or buoyancy effects are dominating [14]. It should be remarked that Bénard himself had an incorrect interpretation of which effect was dominating in his original experiments [2]; Rayleigh's [22] subsequent stability analysis also assumed buoyancy-driven convection. Only several decades later were Bénard experiments correctly interpreted through surface-tension-gradient effects [4,21,23].

To our knowledge, the deflection of the free surface has never before been investigated using simulation tools; only a fixed and flat "free surface" has been used in earlier numerical studies [19,24]. We assume that this is partially due to the fact that the free surface deformation is small, but also partially because the imposition of the simultaneous curvature and surface-gradient effects along curved boundaries is a non-trivial task.

Our goal with this study is to present a way to accurately simulate Bénard-Marangoni convection, including the free surface deformation. Our work will focus on the prediction of steady state solutions. The solutions (velocity, temperature, pressure, and geometry) are expected to be of high regularity, and thus high order spatial discretizations should be very attractive to use for this class of application.

The governing equations for this problem are the incompressible Navier-Stokes equations coupled to a convection-diffusion equation for the temperature. Because our goal is to impose general free surface boundary conditions on potentially deformed surfaces, we need to use the full stress formulation of the Navier-Stokes equations. This formulation is not commonly used in earlier analysis due to the simplified assumptions about a flat "free surface." In addition, similar to the approach presented in [12], we would like to represent the free surface boundary conditions in general curvilinear coordinates; this will prove very advantageous for the subsequent numerical treatment.

An outline of the paper is as follows. In Section 2 we present a derivation of the full mathematical model for Bénard-Marangoni convection. We present the governing equations in strong form, including all the boundary conditions, and allowing for all the possible effects discussed above. Since such a complete model is not commonly used, we present the derivation in some detail, including some important results from differential geometry. We also present the non-dimensionalization of the governing equations and introduce the relevant non-dimensional numbers for this problem. In Section 3 we present the weak formulation of the governing equations. In particular, we exploit the