

## Laminated Wave Turbulence: Generic Algorithms II

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**Abstract.** The model of laminated wave turbulence puts forth a novel computational problem – construction of fast algorithms for finding exact solutions of Diophantine equations in integers of order  $10^{12}$  and more. The equations to be solved in integers are resonant conditions for nonlinearly interacting waves and their form is defined by the wave dispersion. It is established that for the most common dispersion as an arbitrary function of a wave-vector length two different generic algorithms are necessary: (1) one-class-case algorithm for waves interacting through scales, and (2) two-class-case algorithm for waves interacting through phases. In our previous paper we described the one-class-case generic algorithm and in our present paper we present the two-class-case generic algorithm.

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**Key words:** Laminated wave turbulence, discrete wave systems, computations in integers, transcendental algebraic equations, complexity of algorithm.

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### 1 Introduction

The theory of nonlinear dispersive waves begins with hydrodynamics of the 19th century when it was first established that for nonlinear waves dispersive effects might be more important than dissipative. The role of the nonlinear dispersive PDEs in the theoretical physics is determined by their appearance in numerous applications (hydrodynamics, plasma physics, meteorology, etc.) and is so important that the very notion of dispersion is used in physics as a base for classification of all evolutionary PDEs dividing them into two classes – dispersive and non-dispersive [1]. Simply speaking, any evolutionary

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nonlinear PDE (NPDE) is called dispersive if its *linear part* has wave-like solutions of the form

$$\varphi = A \exp i[\vec{k}\vec{x} - \omega t], \quad \omega: \frac{d^2\omega}{dk^2} \neq 0,$$

where  $\vec{k}$  is called wave vector,  $\vec{x}$  - space variable,  $t$  - time variable,  $\omega$  - dispersion function and wave amplitude  $A$  may depend on space variables but not on time  $t$ . Many known integrable systems have this form, for instance, Korteweg-de Vries equation, Kadomtsev-Petviashvili equation, etc. But, of course, most evolutionary PDEs are not integrable and that was the reason why the method of kinetic equation has been developed beginning in 1960's and applied to many different types of dispersive evolutionary PDEs [2–4]. The wave kinetic equation is approximately equivalent to the initial nonlinear PDE for it is an averaged equation imposed on a certain set of correlation functions. Some statistical assumptions have been used in order to obtain kinetic equations; the limit of their applicability then is a very complicated problem which should be solved separately for each specific equation. One of the most important assumptions justifying this approach is the existence of a small parameter  $0 < \varepsilon \ll 1$  which in general defines an upper bound of the magnitudes of the wave amplitudes in such a system. This is the reason why this theory is also called wave turbulence theory – in contrast to fully developed turbulence where a NPDE might not even have a linear part.

The most general problem setting of the wave turbulence theory can be regarded in the form of a nonlinear partial differential equation

$$\mathcal{L}(\psi) = \varepsilon \mathcal{N}(\psi) \quad (1.1)$$

where  $\mathcal{L}$  and  $\mathcal{N}$  denote linear and nonlinear part of the equation correspondingly. Obviously, sum of any linear waves providing solutions of  $\mathcal{L}(\varphi) = 0$  is also a linear wave with a constant amplitude. Intuitively natural expectation is that solutions of Eq. (1.1) with a small nonlinearity will have the same form as linear waves but perhaps with amplitudes "slightly" depending on time. Existence of a small parameter  $\varepsilon$  allows us to use standard multi-scale method [5, 11] and to introduce so-called "slow" time scale,  $T = t/\varepsilon$ , such that now  $A_i = A_i(T)$ . Solution of (1.1) is now looked for in the form

$$\psi = \psi_0(\vec{x}, t, T) + \varepsilon \psi_1(\vec{x}, t, T) + \varepsilon^2 \psi_2(\vec{x}, t, T) + \dots$$

and after substituting this expression into Eq. (1.1) all terms but resonant are neglected. Resonance conditions take following general form

$$\begin{cases} \omega(\vec{k}_1) \pm \omega(\vec{k}_2) \pm \dots \pm \omega(\vec{k}_r) = 0, \\ \vec{k}_1 \pm \vec{k}_2 \pm \dots \pm \vec{k}_r = 0, \end{cases} \quad (1.2)$$

for  $r$  interacting waves with wave-vectors  $\vec{k}_i$ ,  $i = 1, 2, \dots, r$ . The dispersion function  $\omega = \omega(\vec{k})$  can be easily found by substitution of  $\varphi$  into the linear part of the given PDE,  $\mathcal{L}(\varphi) =$