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Numerical Simulation of Fluid Membranes in Two-Dimensional Space

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Abstract. The membrane's dynamics is very important for cells. A membrane in 2-dimensional space can be seen as an incompressible closed curve in a plane or a cylindrical surface in 3-dimensional space. In this paper, we design a second-order accurate numerical algorithm to simulate the shape transformation of the membrane. In the algorithm, we use the tangent angles to present the curve and avoid the difficulties from the constraint of curve's incompressible condition. A lot of interesting phenomena are obtained. Some of them are very like the life processes of cells, such as exocytosis and endocytosis. Furthermore, we can see the relation between two dynamic models clearly. At last, considering the influence of the inner incompressible fluids partially, we add a constraint: the area circled by the membrane maintain invariable. The numerical results show the dynamic motions of a curve remaining its local arc length and inner area constant.

AMS subject classifications: 65M06, 65M12, 92C17

Key words: Director model of the membrane, reduced model, osmotic pressure, spontaneous curvature, constraint of the area.

1 Introduction

The membrane is probably the most important component in the cell. It surrounds all living cells and their organelles to maintain the cell's shape and regulate transport in and out of cells or subcellular domains. It also plays major role in many vital actions of cells, such as segmentation.

Recently, there has been a lot of experimental and analytic research on the configuration and deformation of elastic bio-membranes. In nature, a membrane consists chiefly

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of lipids, proteins and carbohydrates. The structures and properties of membranes are very complex. One common method to simplify the structural analysis is to consider the membrane bilayers formed by certain amphiphilic molecules dissolved in water. In 1973, Helfrich [2] recognized that this lipid bilayer has the structure of a smectic liquid crystal. Based on the elastic theory of liquid crystals, he discovered the curvature elasticity model

$$E_H = \int_\Gamma (a+b(H-c_0)+cG)^2\, ds,$$

where a is the surface tension, b,c are the bending rigidities and c_0 is the spontaneous curvature which describes the asymmetry effect of the membrane or the environment. A lot of work has been done in modeling the membrane using the theories for elastic shells. Steigmann [5, 6] considered the fluid films with curvature elasticity without viscous effects. Waxman [7–9] developed a kinetic model of the fluid dynamics on an evolving surface. Cai and Lubensky [10] derived a system of hydrodynamical equations for a fluid membrane and considered the renormalization of the compressibility and the dissipative coefficients. Pozrikidis [11–16] developed Waxman's model [9]. He considered the membrane as a compressible shell with bending resistance. Miao and his copartners [26–28] presented a general and systematic theory of non-equilibrium dynamics of multi-component fluid membranes. Hu et al. [1] developed an elastic energy model based on the Frank energy of the smectic liquid crystal by introducing the director field. The energy of the lipid directors balances the tendency to point parallel with neighbors and the normal vectors of the surface. If all the directors are constrained artificially on the direction of normal vectors, the energy will be reduced to Helfrich's curvature elastic energy as shown above. When the elastic coefficient in the director model tends to infinity, they obtained a reduced model. This reduced model is very like Waxman's model [9], but adds one term to the in-plane stresses, so that the model satisfy the second law of thermodynamics.

During the past several decades, a lot of numerical simulations have been performed so far. In 1976, Deuling and Helfrich [3] explained the characteristic bi-concave disk-like shape of the resting red blood cell, and obtained a rich catalog of axis-symmetric vesicle shapes with spontaneous curvature by the curvature elastic energy. In 1989, Svetina and Zeks [17] combined the bending elasticity with the bilayer-coupling hypotheses which leads to an additional constraint. They investigated part of the corresponding phase diagram of the equilibrium states. In 1991, Seifert et al. [19] systematically studied axisymmetric shapes which minimize the bending energy and determined the phase diagram for both the spontaneous-curvature and the bilayer-coupling models. They found a new branch of shapes, pear-shaped vesicles, and spheres connected by narrow necks. The occurrence of these shapes is intimately related to the budding phenomenon. In 2002, Capovilla [18] discussed the use of the stress tensor as a basis for perturbation theory and derived the first integral of the shape equation for axisymmetric configurations by examining the forces which are balanced along the circles of constant latitude. In 2004, Gerald and Michael [20] used continuum mechanics to model the red blood cell's membrane