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Computation of High Frequency Wave Diffraction by a Half Plane via the Liouville Equation and Geometric Theory of Diffraction[†]

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Abstract. We construct a numerical scheme based on the Liouville equation of geometric optics coupled with the Geometric Theory of Diffraction (GTD) to simulate the high frequency linear waves diffracted by a half plane. We first introduce a condition, based on the GTD theory, at the vertex of the half plane to account for the diffractions, and then build in this condition as well as the reflection boundary condition into the numerical flux of the geometrical optics Liouville equation. Numerical experiments are used to verify the validity and accuracy of this new Eulerian numerical method which is able to capture the moments of high frequency and diffracted waves without fully resolving the high frequency numerically.

AMS subject classifications: 35L05, 65M06, 78A05, 78A45

Key words: High frequency waves, Liouville equation, geometric theory of diffraction, geometric optics.

1 Introduction

In this paper, we construct a numerical scheme for the high frequency wave equation in two-dimension:

$$u_{tt}-c(\mathbf{x})^2\Delta u=0,\quad t>0, \tag{1.1}$$

$$u(0)=A(\mathbf{x},0)e^{i\phi(\mathbf{x},0)/\epsilon}, \tag{1.2}$$

$$\frac{\partial u}{\partial t}(0)=B(\mathbf{x},0)e^{i\phi(\mathbf{x},0)/\epsilon}, \tag{1.3}$$

[†]Dedicated to Professor Xiantu He on the occasion of his 70th birthday.

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here $c(\mathbf{x})$ is the local wave speed and $\epsilon \ll 1$. When the *essential frequencies* in the wave field are relatively high, and thus the wavelength is short compared to the size of the computational domain, direct simulation of the standard wave equation will be very costly, and approximate models for wave propagation based on geometric optics (GO) are usually used [9,12].

We are concerned with the case when there are some wedges in the computational domain, i.e. the tips and discontinuity in the boundary. When waves hit the wedges, there will be reflections and diffractions.

One of the approximate models for high frequency wave equation is the Liouville equation, which arises in phase space description of geometric optics (GO) [9,32]:

$$f_t + H_{\mathbf{v}} \cdot \nabla_{\mathbf{x}} f - H_{\mathbf{x}} \cdot \nabla_{\mathbf{v}} f = 0, \quad t > 0, \quad \mathbf{x},\mathbf{v} \in R^d, \tag{1.4}$$

where the Hamiltonian H possesses the form

$$H(\mathbf{x},\mathbf{v}) = c(\mathbf{x})|\mathbf{v}| = c(\mathbf{x})\sqrt{v_1^2 + v_2^2 + \cdots + v_d^2}, \tag{1.5}$$

$f(t,\mathbf{x},\mathbf{v})$ is the energy density distribution of particles depending on position $\mathbf{x}$, time t and slowness vector $\mathbf{v}$.

The bicharacteristics of this Liouville equation (1.4) satisfies the Hamiltonian systems:

$$\frac{d\mathbf{x}}{dt} = c(\mathbf{x})\frac{\mathbf{v}}{|\mathbf{v}|}, \quad \frac{d\mathbf{v}}{dt} = -c_{\mathbf{x}}|\mathbf{v}|. \tag{1.6}$$

The derivation of GO does not take into account the effects of geometry of the domain and boundary conditions, which give rise to GO solutions that are discontinuous. Diffractions are lost in the infinite frequency approximation such as the Liouville equation. In this case, correction terms can be derived, as done in *Geometric Theory of Diffraction* (GTD) by Keller in [25]. GTD provides a systematic technique for adding diffraction effects to the GO approximations.

The methods for computing the GO solution can be divided into Lagrangian and Eulerian methods.

Lagrangian methods are based on the ODEs (1.6). The simplest Lagrangian method is the ray tracing method where the ODEs in (1.6) together with ODEs for the amplitude are solved directly with numerical methods for ODEs. This approach is very popular in standard free space GO, [6], and the diffractions, [2,8]. The ray tracing method gives the phase and amplitude of a wave along a ray tube, and interpolation must be applied to obtain those quantities everywhere when rays diverge. Such interpolations can be very complicated for diverging rays.

In the last decade, Eulerian methods based on PDEs have been proposed to avoid some of the drawbacks of the ray tracing method [1]. Eulerian methods discretize the PDEs on fixed computational grids to control errors everywhere and there is no need for interpolation. The simplest Eulerian methods solves the eikonal and transport equations