COMMUNICATIONS IN COMPUTATIONAL PHYSICS
Vol. 5, No. 2-4, pp. 391-397

Commun. Comput. Phys.
February 2009

An Iterative Domain Decomposition Algorithm for the Grad(div) Operator

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Received 16 July 2007; Accepted (in revised version) 8 January 2008

Available online 1 August 2008

Abstract. This paper describes an iterative solution technique for partial differential equations involving the **grad**(div) operator, based on a domain decomposition. Iterations are performed to solve the solution on the interface. We identify the transmission relationships through the interface. We relate the approach to a Steklov-Poincaré operator, and we illustrate the performance of technique through some numerical experiments.

AMS subject classifications: 65N55, 65F10

Key words: Domain decomposition, **grad**(div) operator, stable approximation, iterative substructuring method, Steklov-Poincaré operator.

1 Introduction

The purpose of this paper is to take benefit of recent advances in the use of spectral methods for the stable approximation of the **grad**(div) operator in tensorised Cartesian domains to solve a large class of problems involving this operator in more sophisticated domains that can be viewed as unions of tensorised Cartesian domains [2]. More precisely, we want to solve the symmetric linear elliptic boundary value problem : *for a given data f, find* **u** *solution to*

$$-\nabla(\nabla\cdot\mathbf{u})+\alpha^2\mathbf{u}=\mathbf{f},\quad \text{in}\,\Omega, \tag{1.1}$$

$$\mathbf{u}\cdot\mathbf{n}=0,\qquad \text{on}\,\partial\Omega, \tag{1.2}$$

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by a domain decomposition technique using an iterative algorithm between sub-domains in the spirit of the well-known Dirichlet-Neumann algorithm introduced for the Laplacian operator by Quarteroni (see [5] and the references therein). Here, and in the rest of the paper, $\Omega \subset \mathbb{R}^2$ is a bounded open domain with Lipschitzian border and $\mathbf{n}$ denotes the outer unit normal along the boundary. The constant α is given arbitrarily.

The first question we address in Section 2 is the identification of transmission conditions for the vector operator, on the 'skeleton' of the decomposition, that is on the interface between adjacent sub-domains. This is followed in Section 2.1 by the formulation of an iterative substructuring algorithm. In Sections 2.2 and 2.3 we relate the ensuing problem on the skeleton to a Steklov-Poincaré operator and we give some numerical results. Finally Section 3 concludes the paper.

2 A domain decomposition for the grad(div) operator

We assume that the domain Ω is partitioned into a set of non-overlapping and conforming sub-domains $\Omega_i, i=1\cdots,I$ (see [3]) and for simplicity we assume $I=2$. Let $\overline{\Gamma}:=\overline{\Omega_1}\cap\overline{\Omega_2}$ denote the interface between the two sub-domains considered in our analysis and shown on Fig. 1. Γ will be called the skeleton of the decomposition in the sequel of the paper. We shall also assume that Γ is a Lipschitz one-dimensional manifold.

We call $\mathbf{u}_i$ the restriction to sub-domain Ω_i, $i=1,2$, of the solution $\mathbf{u}$ to the problem (1.1)-(1.2), and by $\mathbf{n}_i$ the outward oriented normal vector on $\partial\Omega_i\cap\Gamma$. For convenience we will set $\mathbf{n}=\mathbf{n}_1$.

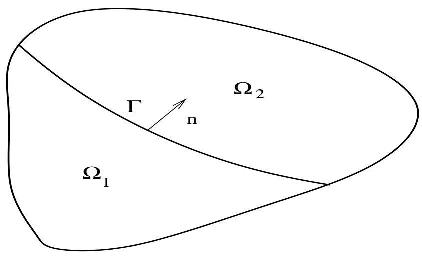

Figure 1: Non-overlapping partition of the domain Ω into two sub-domains.

One can easily prove that the problem (1.1)-(1.2) can be reformulated into the equivalent multi-domain set of local coupled problems (see [5]):

$$-\nabla(\nabla\cdot\mathbf{u}_1)+\alpha^2\mathbf{u}_1=\mathbf{f}, \quad \text{in } \Omega_1, \tag{2.1}$$

$$-\nabla(\nabla\cdot\mathbf{u}_2)+\alpha^2\mathbf{u}_2=\mathbf{f}, \quad \text{in } \Omega_2, \tag{2.2}$$

$$\mathbf{u}_1\cdot\mathbf{n}=0, \quad \text{on } \partial\Omega_1\cap\partial\Omega, \tag{2.3}$$

$$\mathbf{u}_2\cdot\mathbf{n}=0, \quad \text{on } \partial\Omega_2\cap\partial\Omega, \tag{2.4}$$

$$\mathbf{u}_1\cdot\mathbf{n}=\mathbf{u}_2\cdot\mathbf{n}, \quad \text{on } \Gamma, \tag{2.5}$$

$$\mathrm{div}\,\mathbf{u}_1=\mathrm{div}\,\mathbf{u}_2, \quad \text{on } \Gamma. \tag{2.6}$$