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Semi-Implicit Interior Penalty Discontinuous Galerkin Methods for Viscous Compressible Flows

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Abstract. We deal with the numerical solution of the Navier-Stokes equations describing a motion of viscous compressible fluids. In order to obtain a sufficiently stable higher order scheme with respect to the time and space coordinates, we develop a combination of the discontinuous Galerkin finite element (DGFE) method for the space discretization and the backward difference formulae (BDF) for the time discretization. Since the resulting discrete problem leads to a system of nonlinear algebraic equations at each time step, we employ suitable linearizations of inviscid as well as viscous fluxes which give a linear algebraic problem at each time step. Finally, the resulting BDF-DGFE scheme is applied to steady as well as unsteady flows and achieved results are compared with reference data.

AMS subject classifications: 76M10, 76N15, 35Q35, 65L06

Key words: Compressible Navier-Stokes equations, discontinuous Galerkin finite element method, backward difference formulae, linearization.

1 Introduction

Our aim is to develop a sufficiently robust, efficient and accurate numerical scheme for the simulation of unsteady compressible flows. In last years the *discontinuous Galerkin method* (DGM) was employed in many papers for the discretization of compressible fluid flow problems, see, e.g., [5, 6, 8, 10, 20, 32, 37–40, 43–45, 56, 57] and the references cited therein. DGM is based on a piecewise polynomial but discontinuous approximation which provides robust and high-order accurate approximations, particularly in transport dominated regimes. Moreover, there is considerable flexibility in the choice of the

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mesh design; indeed, DGM can easily handle non-matching and non-uniform grids, even anisotropic, polynomial approximation degrees. This allows a simple treatment with *hp*-adaptation techniques. Additionally, orthogonal bases can easily be constructed which lead to diagonal mass matrices; this is particularly advantageous for unsteady problems. Finally, in combination with block-type preconditioners, DGMs can easily be parallelized. For a survey about DGM, see [13] or [15].

There are several variants of the DGM for the solution of problems containing diffusion terms, see, e.g., [3]. It is possible to use a primal formulation or a mixed method. The method can be stabilized with the aid of a symmetric or non-symmetric treatment of diffusion terms, often combined with interior and boundary penalties. The mixed methods consider the gradient of the solution as an independent variable hence the second order derivative in the Navier-Stokes equations are eliminated and consequently, we obtain a problem with a higher number of unknowns, see, e.g., [6]. Nevertheless, an efficient implementation of mixed methods locally eliminates the auxiliary variables. A comparison of accuracy and robustness of the DGM based on the primal formulation from [10] and the mixed DGM from [6] was presented in [9].

Among methods using primal formulation, two approaches, *symmetric interior penalty Galerkin* (SIPG) and *non-symmetric interior penalty Galerkin* (NIPG) introduced in [2] and [50] are very popular, respectively. Moreover, we consider the so-called *incomplete interior penalty Galerkin* (IIPG) method which was studied in [17, 53, 54]. Although IIPG has not the favourable properties as NIPG and SIPG techniques (see Remark 4.3 of this paper), its application to the Navier-Stokes equations is more simple since some stabilization terms are missing. We analyzed these techniques in [25, 27] (NIPG), [24, 26] (SIPG) and [22] (IIPG) for a scalar non-stationary convection-diffusion equation.

For unsteady problems, it is possible to use a discontinuous approximation also for the time discretization (e.g., [46, 56, 57]), but the most usual approach is an application of the method of lines. In this case, the Runge-Kutta methods are very popular for their simplicity and a high order of accuracy, see [6,7,10,14,20,38]. Their drawback is a strong restriction to the size of the time step. To avoid this disadvantage it is suitable to use an implicit time discretization, e.g., [8, 39, 40]. However, a full implicit scheme leads to a necessity to solve a nonlinear system of algebraic equations at each time step which is rather expensive. Therefore, we proposed in [31] a semi-implicit method for the simulation of inviscid compressible flow. This technique is based on a suitable linearization of the Euler fluxes. The linear terms are treated implicitly whereas the nonlinear ones explicitly which leads to a linear algebraic problem at each time step.

In this paper, we extend the approach of semi-implicit scheme to the viscous case. Hence, this article is a natural combination of the explicit scheme for viscous flow from [20] with the semi-implicit scheme for inviscid flow from [31]. Moreover, we apply the *backward difference formula* (BDF) to the time discretization which gives a higher order approximation with respect to the time. This method was analyzed in [29] for the case of a scalar non-stationary convection-diffusion equation.

The content of the rest of the paper is the following. In Section 2 we introduce the