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Multigrid Method for the Chan-Vese Model in Variational Segmentation

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Abstract. The Chan-Vese method of active contours without edges [11] has been used successfully for segmentation of images. As a variational formulation, it involves the solution of a fully nonlinear partial differential equation which is usually solved by using time marching methods with semi-implicit schemes for a parabolic equation; the recent method of additive operator splitting [19, 36] provides an effective acceleration of such schemes for images of moderate size. However to process images of large size, urgent need exists in developing fast multilevel methods. Here we present a multigrid method to solve the Chan-Vese nonlinear elliptic partial differential equation, and demonstrate the fast convergence. We also analyze the smoothing rates of the associated smoothers. Based on our numerical tests, a surprising observation is that our multigrid method is more likely to converge to the global minimizer of the particular non-convex problem than previously unilevel methods which may get stuck at local minimizers. Numerical examples are given to show the expected gain in CPU time and the added advantage of global solutions.

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Key words: Active contours, energy minimization, partial differential equations, segmentation, level sets, total variation, multigrid.

1 Introduction

Image segmentation is a central problem among image processing applications. The aim is to distinguish objects from background and to systematically select specific features out of an image that has many features [2, 10, 22]. For intensity-based images, the

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non-equation-based methods are the popular approaches: threshold techniques, edge-based methods, region-based techniques, and connectivity-preserving relaxation method among others. One may also view the task of distinguishing objects of interest from "the rest" as one to identify the feature boundaries. In recent years, a class of variational formulations offer us the ability to work out features with sharp boundaries — these are the new nonlinear approaches which require more sophisticated solution techniques [10,22].

Let Ω be a bounded open subset of $\mathbb{R}^2$ with $\partial\Omega$ its boundary and let z be the initially given image, which may be a clean image or contain Gaussian noise. Our aim is to extract a desirable image u which represents features within z — more specifically u is piecewise smooth inside each extracted feature.

The purpose of this paper is to present a working multigrid algorithm for implementing the Chan-Vese variational model [11] and to highlight the algorithm's practical advantages.

The rest of the paper is organized in the following way: Section 2 first reviews related variational models and then describes the active contour without edges model by Chan and Vese [11], including a discussion of unilevel solution methods of semi-implicit and additive operator splitting. Section 3 first reviews the nonlinear multigrid framework and then describes our choice of smoothers as well as the multigrid algorithm for solving the underlying differential equation [11]. Section 4 gives some local Fourier analysis of the smoothers used, which forms a basis for multigrid convergence. We end the paper in Section 5 with some numerical results and in Section 6 conclusions.

2 The model of active contour without edges and solution methods

Variational segmentation methods aim to find edges (denoted by the index set Γ below) of features in the image z by directly minimizing some objective functional in order to find the piecewise smooth u function separated by Γ. Different methods choose such functionals differently [10,22]. Two early and related methods are the following.

Firstly, the Mumford and Shah segmentation model [23] finds the desired piecewise smooth (so-called cartoon) image u and the edge set Γ from

$$\min_{u,\Gamma} F_1(u,\Gamma) = \alpha \int_{\Omega\setminus\Gamma} |\nabla u|^2 dxdy + \beta \int_\Gamma d\sigma + \gamma \int_\Omega (u-z)^2 dxdy, \tag{2.1}$$

where α,β,γ are nonnegative constants, the set $\Gamma \subset \Omega$ is also the set of discontinuities, and $\int_\Gamma d\sigma$ is the length of Γ. This minimization is clearly stated but is difficult to implement. Various attempts of approximating this formulation exist.

Secondly, the Ambrosio and Tortorelli model [1] finds u and Γ (via a phase quantity p) from

$$\min_{u,p} F_2(u,p) = \alpha \int_\Omega p^2 |\nabla u|^2 dxdy + \beta \int_\Omega \left(\epsilon |\nabla p|^2 + \frac{(1-p)^2}{4\epsilon} \right) dxdy + \gamma \int_\Omega (u-z)^2 dxdy \tag{2.2}$$