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# Multilevel Preconditioners for the Interior Penalty Discontinuous Galerkin Method II — Quantitative Studies

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**Abstract.** This paper is concerned with preconditioners for interior penalty discontinuous Galerkin discretizations of second-order elliptic boundary value problems. We extend earlier related results in [7] in the following sense. Several concrete realizations of splitting the nonconforming trial spaces into a conforming and (remaining) nonconforming part are identified and shown to give rise to uniformly bounded condition numbers. These asymptotic results are complemented by numerical tests that shed some light on their respective quantitative behavior.

**AMS subject classifications**: 65F10, 65N55, 65N30

**Key words**: Interior penalty method, energy-stable splittings, admissible averaging operators, frames, multilevel Schwarz preconditioners, discontinuous Galerkin methods.

## 1 Introduction

An attractive feature of Discontinuous Galerkin (DG) Finite Element schemes is that this concept offers a unified and versatile discretization platform for various types of partial differential equations. The locality of the trial functions not only supports local mesh refinements but offers also a framework for comfortably varying the order of the discretization. While the error analysis has reached a fairly mature state, less appears to be known about the rigorous foundation of efficient solvers for the linear systems of equations that arise when applying the DG concept to elliptic boundary value problems e.g.

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for locally refined meshes with hanging nodes. In [13] a multigrid scheme was presented and shown to exhibit typical multigrid performance when the solution is sufficiently regular and when the underlying mesh is quasi-uniform. This scheme is extended in [16] to locally refined meshes. While numerical tests indicate that essentially the same efficiency is retained, a theoretical underpinning still seems to be missing. Domain decomposition preconditioners investigated in [1,2], give rise to only moderately growing condition numbers. A two-level scheme in the sense of the "auxiliary space method" (see, e.g., [6,19,24]) has been proposed in [11] and shown to exhibit mesh-independent convergence again on quasi-uniform conforming meshes.

Since the DG concept lends itself to problems whose solutions may exhibit singular behavior we have analyzed in [7] a family of multilevel preconditioners that are shown to give indeed rise to uniformly bounded condition numbers without any additional regularity assumptions and for arbitrary locally refined meshes with hanging nodes (under certain mild grading conditions). While the approach in [7] is primarily based directly on the concept of stable splittings for additive Schwarz schemes [14,18], it can also be interpreted as an "auxiliary space method" in the sense of [6,19,24], a connection that will be detailed below. [7] was primarily concerned with the principal ingredients of a general framework and proposed only a few numerical tests. Aside from commenting on the above mentioned conceptual links the central objective of this paper is to gain additional quantitative information regarding the following issues. A crucial ingredient of our approach is the suitable splitting of the trial space $V_h$ into a conforming and (remaining) nonconforming part, and the key requirements on such splittings where shown to be satisfied in [7] only for the specific case that the conforming part consists of piecewise linear finite elements, representing in some sense the *smallest* conforming subspace contained in $V_h$. Here we shall consider also the *largest* conforming subspace and compare the performance of the respective preconditioners. Moreover, we shall test the robustness of the preconditioners with respect to local mesh refinements. In this context we explore a strategy for adaptive mesh refinements based on [15].

In the remainder of the introduction, we give the precise formulation of the problem, briefly highlight the typical obstructions encountered with the DG method and relate our approach to earlier more abstract results that offer remedies to such obstructions.

### 1.1 Problem formulation

For simplicity we shall confine the discussion to second-order elliptic boundary value problems on polygonal domains $\Omega \subset \mathbb{R}^2$. Our model problem then reads:

$$\text{find } u \in H_0^1(\Omega) \text{ such that} \quad a(u,v) := \langle A\nabla u, \nabla v\rangle + \langle bu, v\rangle = \langle f, v\rangle \quad \forall v \in H_0^1(\Omega), \tag{1.1}$$

where $\langle \cdot,\cdot \rangle$ is the canonical $L_2$-inner product on $\Omega$, $A$ is a (piecewise constant) symmetric positive definite $2\times 2$ matrix, and $b$ a nonnegative (piecewise constant) bounded function on $\Omega$.