Study of Simple Hydrodynamic Solutions with the Two-Relaxation-Times Lattice Boltzmann Scheme

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Abstract. For simple hydrodynamic solutions, where the pressure and the velocity are polynomial functions of the coordinates, exact microscopic solutions are constructed for the two-relaxation-time (TRT) Lattice Boltzmann model with variable forcing and supported by exact boundary schemes. We show how simple numerical and analytical solutions can be interrelated for Dirichlet velocity, pressure and mixed (pressure/tangential velocity) multi-reflection (MR) type schemes. Special care is taken to adapt them for corners, to examine the uniqueness of the obtained steady solutions and staggered invariants, to validate their exact parametrization by the non-dimensional hydrodynamic and a "kinetic" (collision) number. We also present an inlet/outlet "constant mass flux" condition. We show, both analytically and numerically, that the kinetic boundary schemes may result in the appearance of Knudsen layers which are beyond the methodology of the Chapman-Enskog analysis. Time dependent Dirichlet boundary conditions are investigated for pulsatile flow driven by an oscillating pressure drop or forcing. Analytical approximations are constructed in order to extend the pulsatile solution for compressible regimes.

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1   Introduction

Lattice Boltzmann multi-relaxation-time (MRT) models were derived from their Lattice Gas predecessor [7] in pioneering works [19, 20, 30]. These models are simple and efficient, explicit in time, numerical schemes for solving the hydrodynamic Navier-Stokes equations in two [21, 27] and three dimensions [22, 23]. When the collision operator is chosen properly, only a few relaxation parameters are related to the transport coefficients of the derived macroscopic conservation laws, e.g., bulk and kinematic viscosities, and the remaining collision parameters can be viewed as "kinetic" degrees of freedoms. They principally distinguish the method from the direct discretization methods, such as, e.g., finite-difference schemes. Although it was rapidly recognized that the "kinetic" degrees of freedom have a determinant impact on the effective accuracy of microscopic boundary schemes (see [9–11]) and play a significant role for stability [23, 27, 28], the BGK scheme [34] without any kinetic degree of freedom still dominates the modeling of incompressible flow and transport phenomena in porous media. Recently, the MRT model based on the polynomial equilibrium functions [21, 23, 34] attracted more attention for solving complex, single and multiphase problems, e.g., in [29, 31–33, 38–40, 42]. The reader can also find in [37] an exhaustive review on the possibility to increase the stability of the BGK model at low viscosity with the help of an alternative, entropy based, equilibrium.

The main goal of this study is to validate the microscopic solutions and Dirichlet boundary schemes on simple problems with analytical solutions. The multi-reflection type (MR) boundary schemes [16] are constructed in the context of the two-relaxation-time (TRT) linear collision operator [12–15] where they are especially simple and efficient. The TRT operator is suitable for both hydrodynamic and advection-diffusion problems and can be regarded as a bridge between MRT and BGK. TRT shares the simplicity of BGK, but possesses one free collision parameter which plays a crucial role for the overall accuracy and stability, at least for incompressible flow.

We apply the methodology developed in [6, 8, 9, 11, 13, 16], both to construct exact solutions of the MRT/TRT models and to verify the effective accuracy of the boundary schemes. Its key point lies in using the Chapman-Enskog expansion [7, 21] which expresses each individual population in the bulk via its local equilibrium value and its gradients. Based on a parity argument and in the framework of the TRT model, infinite Chapman-Enskog series takes a very simple link-wise form [16], independent of the nature of the equilibrium distribution. When the expected steady solutions for the pressure and velocity are polynomial functions of the coordinates, the expansion has only a finite number of terms. Substituting the truncated, but exact, series into the microscopic boundary rule, we derive the solution for the incoming populations in such problems.

The simplest situation takes place when the derived closure condition, i.e., the difference between the expected one from the Chapman-Enskog analysis and the one obtained from the incoming population, fits the directional Taylor expansion with respect to the local equilibrium component (e.g., for velocity or pressure). The coefficients of