REVIEW ARTICLE

Mathematical Models and Numerical Methods for High Frequency Waves

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Abstract. The numerical approximation of high frequency wave propagation is important in many applications. Examples include the simulation of seismic, acoustic, optical waves and microwaves. When the frequency of the waves is high, this is a difficult multiscale problem. The wavelength is short compared to the overall size of the computational domain and direct simulation using the standard wave equations is very expensive. Fortunately, there are computationally much less costly models, that are good approximations of many wave equations precisely for very high frequencies. Even for linear wave equations these models are often nonlinear. The goal of this paper is to review such mathematical models for high frequency waves, and to survey numerical methods used in simulations. We focus on the geometrical optics approximation which describes the infinite frequency limit of wave equations. We will also discuss finite frequency corrections and some other models.

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1 Introduction

In this review we consider numerical simulation of waves at high frequencies, and the underlying mathematical models used. For simplicity we will mainly discuss the linear scalar wave equation,

$$u_{tt} - c(\boldsymbol{x})^2 \Delta u = 0, \qquad (t,\boldsymbol{x}) \in \mathbb{R}^+ \times \Omega, \qquad \Omega \subset \mathbb{R}^d, \tag{1.1}$$

where $c(\boldsymbol{x})$ is the local speed of wave propagation of the medium. We complement (1.1) with initial or boundary data that generate high-frequency solutions. The exact form of the data will not be important here, but a typical example would be $u(0,\boldsymbol{x}) = A(\boldsymbol{x}) \exp(i\omega \boldsymbol{k} \cdot \boldsymbol{x})$ where $|\boldsymbol{k}| = 1$ and the frequency $\omega \gg 1$. With slight modifications, the techniques we describe will also carry over to systems of wave equations, like the Maxwell equations and the elastic wave equation.

When the frequency of the waves is high, (1.1) is a multiscale problem, where the small scale is given by the wavelength, and the large scale corresponds to the overall size of the computational domain. In the direct numerical simulation of (1.1) the accuracy of the solution is determined by the number of grid points or elements per wavelength. The computational cost to maintain constant accuracy grows algebraically with the frequency, and for sufficiently high frequencies, direct numerical simulation is no longer feasible. Numerical methods based on approximations of (1.1) are needed.

Let us mention before continuing that this multiscale problem is prevalent in many applications for different types of waves: elastic, electromagnetic as well as acoustic. Seismic wave propagation, for instance, is a challenging elastic wave problem. Both the forward and the inverse problems are of great interest and high frequency approximations must be used when the relative wavelength is short. In computational electromagnetics (CEM) radiation and scattering problems, such as radar cross section (RCS) computations, are important. Electromagnetic waves emitted by communication or radar devices often have a very small wavelength compared to the size of the scatterer, which can be an entire aircraft. For acoustic problems high frequency techniques become interesting, for instance, in underwater acoustics where waves of moderate frequency travel over very large distances.

Fortunately, there exist good approximations of many wave equations precisely for very high frequency solutions. In this paper we mainly consider variants of *geometrical optics*, which are asymptotic approximations obtained when the frequency tends to infinity. These approximations are widely used in applications. Instead of the oscillating wave field the unknowns in standard geometrical optics are the phase and the amplitude, which typically vary on a much coarser scale than the full solution. Hence, they should in principle be easier to compute numerically.

Geometrical optics can be formulated in several different ways. Assuming the solution can be approximated by a simple wave,

$$u(t,\boldsymbol{x}) \approx A(t,\boldsymbol{x}) e^{i\omega \phi(t,\boldsymbol{x})}, \tag{1.2}$$