Local Discontinuous-Galerkin Schemes for Model Problems in Phase Transition Theory

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Abstract. Local Discontinuous Galerkin (LDG) schemes in the sense of [5] are a flexible numerical tool to approximate solutions of nonlinear convection problems with complicated dissipative terms. Such terms frequently appear in evolution equations which describe the dynamics of phase changes in e.g. liquid-vapour mixtures or in elastic solids. We report on results for one-dimensional model problems with dissipative terms including third-order and convolution operators. Cell entropy inequalities and L^2-stability results are proved for those model problems. As is common in phase transition theory the solution structure sensitively depends on the coupling parameter between viscosity and capillarity. To avoid spurious solutions due to the counteracting effect of artificial dissipation by the numerical flux and the actual dissipation terms we introduce Tadmors' entropy conservative fluxes. Various numerical experiments underline the reliability of our approach and also illustrate interesting and (partly) new phase transition phenomena.

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1 Introduction

As a basic model problem we consider the initial value problem

$$u_t^\varepsilon + f(u^\varepsilon)_x = R^\varepsilon[u^\varepsilon] \quad \text{in } \Omega_T := \mathbb{R} \times (0,T), \ T > 0, \tag{1.1}$$

$$u^\varepsilon(.,0) = u_0 \quad \text{in } \mathbb{R}. \tag{1.2}$$

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Here, for $\varepsilon > 0$, the unknown function is $u^\varepsilon : \mathbb{R} \times [0,T) \to \mathbb{R}$. By $f \in C^1(\mathbb{R}, \mathbb{R})$ we denote the given flux function and by $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ the initial function. Let us assume that (1.1)-(1.2) is uniquely solvable in an appropriate function space where R^ε is a dissipative operator acting on this space. Specific examples are given below.

We are interested in choices of R^ε such that

$$\lim_{\varepsilon \to 0} R^\varepsilon[w] \equiv 0 \tag{1.3}$$

holds for all functions $w : \mathbb{R} \to \mathbb{R}$ in the function space at hand in the sense of distributions. Then (1.1) turns in the limit $\varepsilon \to 0$ into the hyperbolic equation

$$u_t + f(u)_x = 0 \quad \text{in } \Omega_T. \tag{1.4}$$

Solutions of initial value problems for (1.4) might contain discontinuous shock waves so that one has to consider weak solutions which, however, are not uniquely determined. In this framework it is natural to enforce uniqueness by selecting the admissible weak solution for (1.4) as the function $u : \mathbb{R} \times [0,T) \to \mathbb{R}$ with

$$\lim_{\varepsilon \to 0} \left\| u^\varepsilon - u \right\|_{L^p_{loc}(\Omega_T)} = 0, \tag{1.5}$$

provided the latter limit exists for some $p \geq 1$ and u is a weak solution of (1.4).

For small but positive $\varepsilon > 0$ in (1.1) it is a challenge to solve the initial value problem numerically since then the solution is governed by the behaviour of the limit problem and can contain steep internal layers. Additionally the numerical entropy dissipation has to be tuned very carefully since the limit (1.5) can sensitively depend on the structure of R^ε as we shall detail below. The Local Discontinuous Galerkin (LDG)-scheme provides an elegant and flexible tool to treat quite general versions of (1.1), in particular the (formal) order of the method can be chosen without restrictions. The approach has been originally introduced in [5] for diffusion operators and since then has been applied to many other evolution equations so that we only refer to the overview publications [2, 3]. The LDG-approach relies on a reformulation of (1.1) as a degenerate first-order system and the discretization of this system by the (classical) Discontinuous-Galerkin method (cf. [4]) for first-order systems. We note also that the LDG-scheme requires to use numerical flux functions to discretize the term $f(u^\varepsilon)_x$ and the dissipative fluxes that come out of R^ε in (1.1).

In this paper we test the LDG-scheme for complex choices for R^ε which have been recently suggested as models for phase transition phenomena. We are interested in cases where the limit in (1.5) exists but leads to non-standard weak solutions (i.e., weak solutions which not necessarily are Kruzkov-solutions) of (1.4).

A well analyzed choice for R^ε in (1.1) such that u from (1.5) exists is

$$R^\varepsilon[w] = \varepsilon w_{xx}, \quad w \in C^2(\mathbb{R}). \tag{1.6}$$