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A Family of Relaxation Schemes for Nonlinear Convection Diffusion Problems

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Abstract. In this work we present a family of relaxation schemes for nonlinear convection diffusion problems, which can tackle also the cases of degenerate diffusion and of convection dominated regimes. The schemes proposed can achieve any order of accuracy, give non-oscillatory solutions even in the presence of singularities and their structure depends only weakly on the particular PDE being integrated. One and two dimensional results are shown, and a nonlinear stability estimate is given.

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Key words: Convection-diffusion, relaxation schemes, degenerate parabolic problems.

1 Introduction

Relaxation approximations to non-linear PDE's are based on the replacement of the original PDE with a semi-linear hyperbolic system of equations, with a stiff source term, tuned by a relaxation parameter ε. When $\varepsilon \to 0$, the system relaxes onto the original PDE. A consistent discretization of the relaxation system for $\varepsilon = 0$ yields a consistent discretization of the original PDE, see for instance [2,4,9]. The advantage of this procedure is that the numerical scheme obtained in this fashion does not need computationally expensive Riemann solvers for the non-linear convective term, but enjoys the robustness of upwind

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discretizations, since the convective term is replaced by a constant coefficient linear hyperbolic system to which upwinding can be easily applied.

Moreover the complexity introduced replacing the original PDE with a stiff system of equations is only apparent, because when the discretization is carried out in the relaxed case, one recovers a scheme where only the original unknown is actually updated.

In this work, we consider convection-diffusion equations of the form:

$$\frac{\partial u}{\partial t}+\operatorname{div} f(u)=\Delta(p(u)), \tag{1.1}$$

where f is a differentiable function of the form $f(u)=[f_1,\cdots,f_d]$, with d denoting the number of space dimensions.

We assume that all derivatives f_j', $1\le j\le d$, in absolute value are bounded by a constant α; thus α is an upper bound of the maximum characteristic speed of the equation, measured in a suitable norm.

The function $p(u)$ appearing in the diffusion term is a non-negative, non-decreasing, Lipshitz continuous function with Lipshitz constant δ. The equation is called degenerate when (i) the (weak) derivative; (ii) $p'(u)$ vanishes for some nonnegative values of u: in this case, even the parabolic term generates travelling fronts. Here we illustrate the case of a scalar equation, but the construction can be easily generalized to systems.

Note also that analogies can be found between the relaxation approach and the LDG method (Local Discontinuous Galerkin) for convection diffusion equations [7]. In both cases the order of the PDE is reduced introducing an auxiliary equation. Furthermore, the relaxation system, unlike LDG, also linearizes the system with the help of an auxiliary variable. The linearized system provides a natural way to construct numerical fluxes (through upwinding) which results in a scheme which is stable under a CFL-like condition. On the other hand, LDG requires the construction of numerical fluxes and stabilization terms, so that stability is not built into the system being discretized. In the purely convective case the relaxation system utilizes the full technology of hyperbolic nonoscillatory reconstructions to prevent the onset of spurious oscillations.

Several strategies are possible to introduce relaxation systems for equation (1.1), see also [2]. We will compare the numerical properties of a few classes of relaxation approaches in a forthcoming paper. Here we propose a simple strategy which results in a particularly efficient scheme. It is interesting to note that the resulting scheme cannot be cast in the BGK framework studied in [2].

The present paper starts with a detailed description of the one-dimensional case, computing a stability estimate for the first order scheme, which will guide the selection of the time step even for higher order schemes. Next we discuss the implementation of boundary conditions and the extension of the scheme to the multidimensional case. We end with a few test problems carried out for schemes of order from 2 to 5, demonstrating the high resolution and non oscillatory properties of the family of schemes we propose.