

## Numerical Investigations of the Dynamical Behaviors and Instabilities for the Gierer-Meinhardt System

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**Abstract.** This work is concerned with the numerical simulations on the Gierer-Meinhardt activator-inhibitor models. We consider the case when the inhibitor time constant  $\tau$  is non-zero. In this case, oscillations and pulse splitting are observed numerically. Numerical experiments are carried out to investigate the dynamical behaviors and instabilities of the spike patterns. The numerical schemes used are based upon an efficient moving mesh finite element method which distributes more grid points near the localized spike regions.

**AMS subject classifications:** 65M50, 65M60

**Key words:** Gierer-Meinhardt model, moving finite element method, pulse splitting, spikes.

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### 1 Introduction

The generation of spatial pattern of tissue structures is one of the elementary processes in morphogenesis. Since the pioneering work of Turing [3] in 1952, there have been many studies on two-component reaction-diffusion systems for the formation of spatially complex patterns, see [4, 8, 9, 31]. It has been largely explored that the instabilities of spatially homogeneous patterns can develop varying states via the mechanism of local self-enhancement and long range inhibition. However, in the singularly perturbed limit, many reaction-diffusion systems can give rise to spike-type patterns whereby one of the components of the system becomes spatially localized at certain regions in the domain. In contrast to spatially homogeneous solutions, the instabilities and the dynamics of these localized patterns are not nearly as well understood.

Various models have been proposed for the pattern formation, such as Gray-Scott model [6], Schnakenberg model [7] and Gierer-Meinhardt (GM) model [5]. Among

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these models, GM model seems to be the most well-known reaction-diffusion system of activator-inhibitor type. It has been widely used to model localization processes in nature, such as cell differentiation and morphogenesis [1,31], and the formation of sea-shell patterns [2]. The dimensionless GM model can be written as (cf. [10,15])

$$a_t = \epsilon^2 \Delta a - [1 + V(x)]a + \frac{a^p}{h^q}, \quad x \in \Omega, \quad t > 0, \quad (1.1)$$

$$\tau h_t = D \Delta h - \mu h + \epsilon^{-N} \frac{a^r}{h^s}, \quad x \in \Omega, \quad t > 0, \quad (1.2)$$

$$\partial_n a = \partial_n h = 0, \quad x \in \partial \Omega, \quad (1.3)$$

$$a(x, y, 0) = a^0(x, y), \quad h(x, y, 0) = h^0(x, y). \quad (1.4)$$

Here  $a, h$ ,  $0 < \epsilon \ll 1$ ,  $D > 0$ ,  $\mu > 0$ ,  $V = V(x) \geq 0$  and  $\tau \geq 0$ , represent the scaled activator concentration, inhibitor concentration, activator diffusivity, inhibitor diffusivity, inhibitor decay rate, activator decay rate, and inhibitor time constant, respectively.  $\partial_n$  is the outward normal derivative to the boundary, and  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ . The exponents  $(p, q, r, s)$  are assumed to satisfy

$$p > 1, \quad q > 0, \quad r > 1, \quad s \geq 0, \quad \frac{p-1}{q} < \frac{r}{s+1}. \quad (1.5)$$

For  $\epsilon \ll 1$ , many studies of GM model have shown the spike patterns become narrower and narrower when  $\epsilon \rightarrow 0$ , see [1,24]. In fact, their spatial extension is of order  $\mathcal{O}(\epsilon)$ . And the spike patterns also have various dynamical behaviors, such as the drift of the center of the spikes, the oscillation of the height of the spikes, even the splitting of the spikes. So very fine meshes over the spatial extension of the spikes are needed to resolve this problem. Although using a very large number of equidistantly spaced spatial mesh points to solve the spike dynamics is possible in one space dimension, it is computationally inefficient. And for multi space dimensions, it is almost infeasible to simulate multi-spikes dynamics using uniform mesh approach. In viewing of this, some adaptive grid strategies should be applied. In [14], satisfactory numerical solutions for the one-dimensional GM and Schnakenberg model are obtained using the moving mesh method. Moreover, there are also several other works on one-dimensional GM model, see, e.g., [10,11,25,26]. For two-dimensional GM model, some analysis works can be found in [15,23,24,32]. However, because of the extremely large computational cost, there are only a few works in numerical simulations for spike dynamics in 2D, see [10,22,31].

Numerical difficulties in simulating two-dimensional GM model also lie in that there are different orders of errors: the error in spike height is  $\mathcal{O}(\frac{1}{\log \frac{1}{\epsilon}})$ , the error in spike location is  $\mathcal{O}(\epsilon)$ , the critical threshold for  $D$  is  $\mathcal{O}(\log \frac{1}{\epsilon})$  and the time evolution for spikes is  $\mathcal{O}(\frac{1}{\epsilon^2 \log \frac{1}{\epsilon}})$ . The traditional finite element method (FEM) can not resolve the spike dynamics for very small  $\epsilon$ .

In this paper, we use moving finite element method, introduced in [12,13], to simulate the spike dynamics of the singularity perturbed Gierer-Meinhardt (GM) model in two