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REVIEW ARTICLE

Some Recent Advances on Spectral Methods for Unbounded Domains

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Abstract. We present in this paper a unified framework for analyzing the spectral methods in unbounded domains using mapped Jacobi, Laguerre and Hermite functions. A detailed comparison of the convergence rates of these spectral methods for solutions with typical decay behaviors is carried out, both theoretically and computationally. A brief review on some of the recent advances in the spectral methods for unbounded domains is also presented.

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Contents

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1 Introduction

Spectral methods for solving PDEs on unbounded domains can be essentially classified into four approaches:

(i) Domain truncation: truncate unbounded domains to bounded domains and solve the PDEs on bounded domains supplemented with artificial or transparent boundary conditions (see, e.g., [17,21,22,25,44,51]);

(ii) Approximation by classical orthogonal systems on unbounded domains, e.g., Laguerre or Hermite polynomials/functions (see, e.g., [7,14,20,30,31,36,43,47]);

(iii) Approximation by other, non-classical orthogonal systems (see, e.g., [14]), or by mapped orthogonal systems, e.g., image of classical Jacobi polynomials though a suitable mapping (see, e.g. [32,34,35,54]);

(iv) Mapping: map unbounded domains to bounded domains and use standard spectral methods to solve the mapped PDEs in the bounded domains (see, e.g., [9–12,15,24,26]).

Boyd provided in [11] an excellent review on general properties and practical implementations for many of these approaches. In general, the domain truncation approach is only a viable option for problems with rapidly (exponentially) decaying solutions or when accurate non-reflecting or exact boundary conditions are available at the truncated boundary. On the other hand, with proper choices of mappings and/or scaling parameters, the other three approaches can all be effectively applied to a variety of problems with rapid or slow decaying (or even growing) solutions. Since there is a vast literature on domain truncations, particularly for Helmholtz equations and Maxwell equations for scattering problems and the analysis involved is very different from the other three approaches, the domain truncation approach will not be addressed in this paper.

We note that the last two approaches are mathematically equivalent (see Section 2.5.1 for more details) but their computational implementations are different. More precisely, the last approach involves solving the mapped PDEs (which are often cumbersome to deal with) using classical Jacobi polynomials while the third approach solves the original PDE using the mapped Jacobi polynomials. The main advantage of the last approach is that it can be implemented and analyzed using standard procedures and approximation results, but its main disadvantage is that the transformed equation is usually very complicated which, in many cases, makes its implementation and analysis unusually cumbersome. On the other hand, we work on the original PDE in the third approach and approximate its solution by using a new family of orthogonal functions which are images of classical Jacobi polynomials under a suitable mapping. The analysis of this approach will require approximation results by the new family of orthogonal functions. The main advantage is that once these approximation results are established, they can be directly applied to a large class of problems. Thus, we shall mainly concentrate on the second and third approaches, and provide a general framework for the analysis of these spectral methods.

While spectral methods have been used for solving PDEs on unbounded domains