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Effectiveness of Implicit Methods for Stiff Stochastic Differential Equations

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Abstract. In this paper we study the behavior of a family of implicit numerical methods applied to stochastic differential equations with multiple time scales. We show by a combination of analytical arguments and numerical examples that implicit methods in general fail to capture the effective dynamics at the slow time scale. This is due to the fact that such implicit methods cannot correctly capture non-Dirac invariant distributions when the time step size is much larger than the relaxation time of the system.

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1 Introduction

Implicit stiff ODE solvers have been very successful and have become the method of choice for a large class of stiff ODEs [7]. In a system for which different components evolve on different time scales, these methods allow us to capture the dynamics of the system on the slow time scale without resolving the transient effects on the fast time scale.

Most problems for which stiff ODE solvers have been successful are those for which the trajectories reach a stable manifold after a possible initial transient corresponding to a fast scale. A convenient way to think about these problems is to use the concept of slow manifolds to which the fast variables are attracted. This happens over a short relaxation time scale over which the slow variables can be considered fixed.

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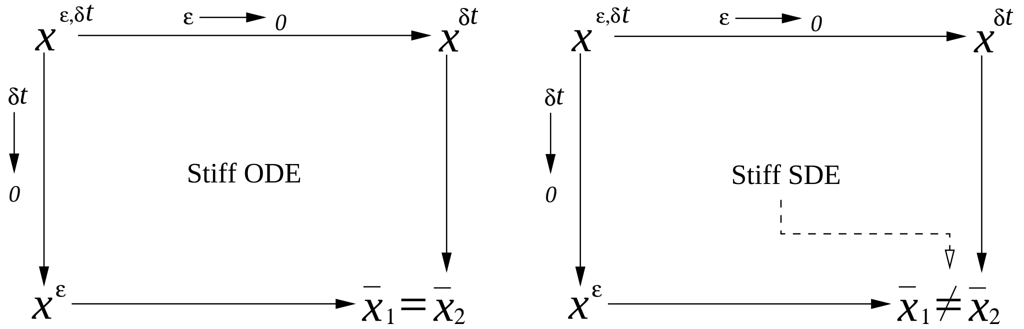

Figure 1: Illustration of the difference between implicit stiff ODE solvers applied to stiff ODEs (left) and SDEs (right).

In this paper we examine the situation when such stiff ODE solvers are applied to stochastic dynamical systems with multiple time scales. We will demonstrate that for such systems, stiff ODE solvers may not be effective when the invariant measure is non-Dirac and produce wrong solutions when the fast scale dynamics are not resolved.

The main point of this paper is shown in Fig. 1. Denote by $x^{\varepsilon,\delta t}$ the numerical solution using an stiff ODE solver with time stepsize δt. ε is a parameter that measures the ratio of the fast and slow time scales in the system. In the case of stiff ODEs of dissipative type, we expect the following to hold:

$$\lim_{\varepsilon\to 0}\lim_{\delta t\to 0} x^{\varepsilon,\delta t} = \lim_{\delta t\to 0}\lim_{\varepsilon\to 0} x^{\varepsilon,\delta t}. \tag{1.1}$$

The right hand side is much less costly to compute and this is at the heart of the effectiveness of these stiff ODE solvers. However, we will demonstrate that for stiff stochastic differential equations (SDEs) in general

$$\lim_{\varepsilon\to 0}\lim_{\delta t\to 0} x^{\varepsilon,\delta t} \neq \lim_{\delta t\to 0}\lim_{\varepsilon\to 0} x^{\varepsilon,\delta t}. \tag{1.2}$$

More precisely, we will see that if we fix the ratio $\delta t/\varepsilon = c$ and we let ε go to zero, we have

$$\lim_{\delta t/\varepsilon = c,\ \varepsilon\to 0} x^{\varepsilon,\delta t} = \bar{x}^c, \tag{1.3}$$

and $\bar{x}^c$ in general varies with c.

2 An illustrative example

Consider the following example of a multiscale ODE:

$$\dot{x} = -y^2 + 5\sin(2\pi t), \tag{2.1}$$

$$\dot{y} = \frac{1}{\varepsilon}(x-y), \tag{2.2}$$