

Bilinear Forms for the Recovery-Based Discontinuous Galerkin Method for Diffusion

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Abstract. The present paper introduces bilinear forms that are equivalent to the recovery-based discontinuous Galerkin formulation introduced by Van Leer in 2005. The recovery method approximates the solution of the diffusion equation in a discontinuous function space, while inter-element coupling is achieved by a local L_2 projection that recovers a smooth continuous function underlying the discontinuous approximation. Here we introduce the concept of a local “recovery polynomial basis” – smooth polynomials that are in the weak sense indistinguishable from the discontinuous basis polynomials – and show it allows us to eliminate the recovery procedure. The recovery method reproduces the symmetric discontinuous Galerkin formulation with additional penalty-like terms depending on the targeted accuracy of the method. We present the unique link between the recovery method and discontinuous Galerkin bilinear forms.

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1 Introduction

Highly accurate schemes for solving the advection equation are preferably obtained by means of the discontinuous Galerkin (DG) method. This method approximates the solution in an element-wise continuous function space that is globally discontinuous. Because of the physics of advection, the method acquires a natural upwind character that renders the discontinuities at the cell interfaces harmless and stabilizes the scheme.

When combining advection with diffusion, though, we run into a problem: the discontinuous function space that works so well for advection does not combine naturally

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with the diffusion operator. In the course of 30 years, various bilinear DG forms have been introduced for approximation of the diffusion operator; the symmetric DG form, stabilized either with the penalty term of Baker [1] (usually attributed to Arnold [2]) or the penalty term of Bassi, Rebay et al. [3], also credited to Brezzi [4], are most widely used.

In constructing bilinear forms for diffusion, the essence lies in choosing the numerical fluxes that are responsible for the coupling of the discontinuous solution approximation across the cell interfaces. Traditionally, these numerical fluxes are defined such that the bilinear form satisfies a number of mathematical conditions (the more the better) such as symmetry, coercivity, boundedness, consistency and adjoint consistency. Until recently, however, an analysis in which bilinear forms for diffusion, with desirable mathematical properties, appear simply as a result of a physical argument, was lacking.

In 2005 Van Leer et al. presented the “recovery method.” Here the coupling through the numerical fluxes is obtained by arguing that, for diffusion, the discontinuous solution approximation should locally be regarded as an L_2 projection of a higher-order continuous function. This acknowledges the physical datum that diffusion produces a smooth solution at any $t > 0$ even from discontinuous initial values. The local “recovered” function couples neighboring cells and provides the information for computing the diffusive fluxes.

In the present paper we consider the 1-D diffusion equation in order to present the link between the recovery method and traditional discontinuous Galerkin bilinear formulations. Key to the systematic derivation of the diffusive fluxes that appear in the bilinear forms is the discovery of the “recovery polynomial basis:” to each piecewise continuous polynomial basis of degree k defined on two adjacent cells corresponds a *unique* continuous polynomial space of degree $2k+1$. In consequence, to the approximation of the solution as an expansion in the discontinuous basis functions locally corresponds an *identical* expansion in the smooth recovery basis; the latter permits computing of the numerical fluxes across the cell interfaces. And, because of the duality of the polynomial spaces, the numerical fluxes in terms of the discontinuous basis functions follow immediately.

Thus, for any polynomial space of degree k the recovery method is equivalent to a unique, basis-independent bilinear discontinuous Galerkin formulation.

The outline of the paper is as follows. In the next section the recovery method is reviewed, in Section 3 the recovery basis is introduced, and in Section 4 the numerical fluxes are computed. These lead to the bilinear forms presented in Section 5. The final section lists the paper’s conclusions.

2 The recovery method

Let us consider the diffusion equation $u_t = Du_{xx}$ which for convenience we discretize on the regular infinite grid

$$\mathbb{Z}_h = \{jh \mid j \in \mathbb{Z}, h > 0\}. \quad (2.1)$$