

## Chebyshev Collocation for Optimal Control in a Thermoconvective Flow

M. C. Navarro<sup>1</sup>, H. Herrero<sup>1,\*</sup> and S. Hoyas<sup>2</sup>

<sup>1</sup> *Departamento de Matemáticas, Facultad de Ciencias Químicas, Universidad de Castilla-La Mancha, 13071 Ciudad Real, Spain.*

<sup>2</sup> *Departamento de Informática, Escuela de Ingeniería, Universidad de Valencia, 46100 Burjassot, Valencia, Spain.*

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**Abstract.** In this paper a Chebyshev collocation method is used for solving numerically an optimal boundary control problem in a thermoconvective fluid flow. The aim of this study is to demonstrate the capabilities of these numerical techniques for handling this kind of problems. As the problem is treated in the primitive variable formulation additional boundary conditions for the pressure and the auxiliary pressure fields are required to avoid spurious modes. A dependence of the convergence of the method on the penalizing parameter that appears in the functional cost is observed. As this parameter approaches zero some singular behaviour in the control function is observed and the order of the method decreases. These singularities are irrelevant in the problem as a regularized control function produces the same results.

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## 1 Introduction

Apart from its appearance in nature, thermoconvective flows occur frequently in industrial applications. For instance, thermoconvective instabilities are responsible for undesirable convective states in some industrial processes such as crystal growth, laser welding or alloy manufacturing [21, 22]. In these processes it is important to avoid convective patterns in order to achieve homogeneous and resistant materials. In other words, the control of fluids for the purpose of achieving some desired objective is crucial in those

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\*Corresponding author. *Email addresses:* MariaCruz.Navarro@uclm.es (M. C. Navarro), Henar.Herrero@uclm.es (H. Herrero), sergio.hoyas@uv.es (S. Hoyas)

applications. In the past, these control problems have been addressed either through expensive experimental processes or through the introduction of significant simplifications into the analysis used in the development of control mechanisms. Recently mathematicians and scientists have been able to address flow control problems in a systematic, rigorous manner and have established a mathematical and numerical foundation for these problems; see [2, 8, 9, 15, 24]. The Chebyshev collocation method is a numerical method broadly used in thermoconvective problems [13, 16, 17]. It has been theoretically studied for fluid dynamics problems [5–7]. But it has not been used in control problems in fluid dynamics because singularities appear in these problems, and spectral methods are less efficient in that case. For this reason finite element approach is the usual method [10, 12].

In this paper, the capabilities of Chebyshev collocation for handling control problems in fluid dynamics will be demonstrated. A Chebyshev collocation method is used to solve numerically a boundary optimal control in a Rayleigh Bénard problem [3, 23] in a cylinder. This problem is extensively described in [18]. The control problem is formulated as a constrained optimization problem, where the constraint is the system of equations that represents steady viscous incompressible Navier-Stokes equations coupled with the energy equation. The choice for the cost is a quadratic functional involving the vorticity in the fluid so that a minimum of that functional corresponds to the minimum possible vorticity subject to the constraints. A linear stability analysis on the controlled states is also performed, so that the convergence tests are performed in terms of the critical values of the bifurcation parameter.

The article is organized as follows. In the second section the convection problem under localized heating is described. The third section explains the optimal control on this problem. The fourth section comprises the numerical results and finally the concluding remarks are summarized in the last section.

## 2 Formulation of the problem

The physical setup considered consists of a horizontal fluid layer in a cylindrical container of radius  $l$  ( $r$  coordinate) and depth  $d$  ( $z$  coordinate). The upper surface is flat and open to the atmosphere where the temperature is  $T_0$ . The bottom plate and lateral walls are rigid and the fluid is heated from below by imposing a Gaussian temperature profile which takes the value  $T_{\max}$  at  $r=0$  and the value  $T_{\min}$  at the outer part ( $r=l$ ).

### 2.1 Equations

The system evolves according to the momentum and mass balance equations and to the energy conservation principle. We concentrate on the study of stationary solutions and, in this sense, the stationary problem is considered. In the equations that govern the system,  $u_x$ ,  $u_y$  and  $u_z$  are the components of the velocity field  $\mathbf{u}$  of the fluid,  $T$  the temperature,  $p$  the pressure and  $\mathbf{x} = (x, y, z)$  are the spatial coordinates. The governing dimen-