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Computing Multivalued Solutions of Pressureless Gas Dynamics by Deterministic Particle Methods

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Abstract. We compute multivalued solutions of one- and two-dimensional pressureless gas dynamics equations by deterministic particle methods. Point values of the computed solutions are to be recovered from their singular particle approximations using some smoothing procedure. We study several recovery strategies and demonstrate ability of the particle methods to achieve high resolution.

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Key words: Pressureless gas dynamics equations, semiclassical transport models, deterministic particle methods.

1 Introduction

We are interested in computing multivalued solutions of the pressureless gas dynamics equations, which, in the two-dimensional (2-D) case, read:

$$\begin{cases} \rho_t+(\rho u)_x+(\rho v)_y=0, \\ (\rho u)_t+(\rho u^2)_x+(\rho uv)_y=-\rho V_x(x,y), \\ (\rho v)_t+(\rho uv)_x+(\rho v^2)_y=-\rho V_y(x,y), \end{cases} \tag{1.1}$$

where ρ is the density, u and v are the x- and y-components of the velocity, respectively, and V is the potential. These equations arise in the modeling of the formation of large scale structures in the universe [30]. They can be formally obtained as the limit of the isotropic Euler equations of gas dynamics as pressure tends to zero or as the macroscopic limit of a Boltzmann equation when the Maxwellian has zero temperature. The most interesting feature of this model is development of strong singularities — delta-shocks

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both at separate points and along shock surfaces. Because of this, mathematical analysis of pressureless gas dynamics equations is quite complicated. We refer the reader to [2,4, 5,13,28] for some recent results.

Capturing delta-shocks numerically is also a challenging problem. Several finite-volume, kinetic, relaxation methods, as well as methods based on the movement of a system of particles have been proposed in the literature (see, e.g., [1,3,10,22] and references therein). One of them is a sticky particle (SP) method recently developed in [10]. Due to its low dissipation nature, the SP method allows one to accurately capture strong singularities as well as to achieve high resolution of the smooth parts of the solution. The main idea of the SP method was to coalesce approaching particles and to average velocities of the particles located in the same cells of the auxiliary grid. This way a computation of a singular single-valued solution was ensured; see [10] for details.

Pressureless gas dynamics equations also arise in semiclassical approximations of oscillating solutions of the Schrödinger equation with the high frequency initial data (a brief derivation of this model is given in Section 2). In this situation, multivalued solutions—not the (singular) single valued ones—of the pressureless gas dynamics equations are physically relevant (see, e.g., [17]). A number of numerical methods have been recently proposed for computing multivalued solutions in different contexts, see, e.g., [14,17–21,23,27] and references therein.

In this paper, we are interested in capturing multivalued solutions of pressureless gas dynamics using non-dissipative particle methods. We note that none of the aforementioned special SP techniques is needed in the model under consideration. This means that we should allow several particles to be located exactly at the same point (representing several branches of the computed solution!) and to propagate with the velocities that are completely independent of the velocities of their neighbors. The resulting particle method is described in Section 3.

One of the major difficulty in the application of particle methods to the pressureless gas dynamics equations is recovery of the point values of the computed solution from its particle approximation. The commonly used approach—approximation of the Dirac delta functions by its convolution with a smooth kernel (see, e.g., [26])—may not properly work in the case of nonsmooth solutions. Recovery of point values of nonsmooth solutions has been studied in [8], where several possible approaches have been discussed (see also [7,9]). Recovery of (single valued) solutions from multivalued particle distributions is even more delicate issue since several solution branches have to be averaged. As we demonstrate in Section 4.1, in the one-dimensional (1-D) case, several techniques lead to high resolution nonoscillatory results. The 2-D case is much more complicated, but we are still able to design a satisfactory solution reconstruction, as shown in Section 4.2.

Another difficulty in the application of the deterministic particle method to multivalued solution computations becomes apparent when thin quantum barriers are present, that is, when the potential V is discontinuous so that the Dirac delta functions appear on the right-hand side (RHS) of (1.1) and (2.3). In this case, we modify the particle method along the lines of [18,19]: a particle that reaches the barrier may pass it with a certain