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p-Multigrid Method for Fekete-Gauss Spectral Element Approximations of Elliptic Problems

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Abstract. An efficient p-multigrid method is developed to solve the algebraic systems which result from the approximation of elliptic problems with the so-called Fekete-Gauss Spectral Element Method, which makes use of the Fekete points of the triangle as interpolation points and of the Gauss points as quadrature points. A multigrid strategy is defined by comparison of different prolongation/restriction operators and coarse grid algebraic systems. The efficiency and robustness of the approach, with respect to the type of boundary condition and to the structured/unstructured nature of the mesh, are highlighted through numerical examples.

AMS subject classifications: 65N30, 65N35, 65N55

Key words: Spectral elements, Fekete points, multigrid methods.

1 Introduction

The Spectral Element Method (SEM), developed in the 80's to solve with spectral-like methods Partial Differential Equations (PDE) in non-Cartesian (non-cylindrical, non-spherical,$\cdots$) geometries, has proved to be very successful during the two last decades, see, *e.g.*, [8, 16]. Its main drawback is however to be not really adapted to very complex geometries, due to the non-simplicial shape of the elements which are the image of the cube $\hat{\Omega}=(-1,1)^d$, where d is the space dimension, in which the polynomial approximation holds.

Some ways have been suggested to support triangular/tetrahedral elements and hence simplicial meshes. Among them is the one proposed in [16], which makes use of the "collapsed coordinate system" resulting from a singular mapping from the

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2D/3D cube onto the triangle/tetrahedron. This approach has appeared of great interest, but suffers from a non-symmetric distribution of the interpolation points in the triangle/tetrahedron, with an useless accumulation of these points in one of the vertices.

The SEM being a nodal method, *i.e.*, the basis functions are Lagrange polynomials based on interpolation points, the main research axis was then to provide points in the simplex showing nice interpolation properties, *i.e.*, such that the Lebesgue constant does not increase fastly with the degree of the polynomial approximation, see, *e.g.*, [4,5,13,14]. Here we are interested in Fekete points based methods, as proposed for the triangle in [27], due to their nice interpolation properties and strong link with the Gauss-Lobatto Legendre (GLL) nodes of the quadrangle based SEM, say QSEM, since Fekete points and GLL points coincide in the d-dimensional cube [2].

In contrast to the GLL points, the Fekete points are however not Gauss points, so results obtained with the earlier triangle based SEM, say TSEM, proposed in [28] may be disappointing. High-accuracy quadrature rules are indeed needed to preserve the "spectral accuracy" of SEM type methods, which are based on variational formulations. This has motivated new researches, to find a unique set of points with nice interpolation and quadrature properties [29, 30] or at least to develop more sophisticated quadrature rules [31]. Such researches are not yet satisfactory. Thus, the quadrature rule of [31] is costly and requires a linear mapping from the reference triangle T to the spectral element; if the mapping is non-linear, then a quadrature rule specific to each element must be set up [15]. For us we have proposed to consider the Gauss points of the triangle as quadrature points and the Fekete points as interpolation points, in the frame of a "Fekete-Gauss TSEM" [20].

Once the approximation procedure is fixed it remains to develop efficient solvers for the associated algebraic systems. As well known, the matrices resulting from high order approximations are indeed ill-conditioned, with $\mathcal{O}(N^4)$ condition numbers in 2D, where $N \equiv p$ is the total degree of the polynomial approximation in each spectral element. We thus have focused on domain decomposition techniques, each spectral element being considered as a subdomain. The following methods have been considered:

- Neumann-Neumann Schur complement methods [21]: Addressing the Schur complement with Neumann-Neumann type preconditioners has yielded promising results. Moreover, the condition number of the Schur complement only shows a $\mathcal{O}(N)$ behavior.
- Overlapping Schwarz methods [22]: Impressive results can be obtained but with the drawback that, in contrast to the QSEM, a "generous overlap" (overlap of one entire mesh element) must be used due to the non-tensorial distribution of the Fekete points in the element.

In parallel, it was of interest to revisit the p-multigrid approach, which makes use of a fixed simplicial mesh and of different approximation levels, each of them associated with a different polynomial degree. For the QSEM this was initially suggested in [18, 23, 24] and recently used in conjunction with Overlapping Schwarz preconditioners for