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A Moving-Mesh Finite Element Method and its Application to the Numerical Solution of Phase-Change Problems

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Abstract. A distributed Lagrangian moving-mesh finite element method is applied to problems involving changes of phase. The algorithm uses a distributed conservation principle to determine nodal mesh velocities, which are then used to move the nodes. The nodal values are obtained from an ALE (Arbitrary Lagrangian-Eulerian) equation, which represents a generalization of the original algorithm presented in *Applied Numerical Mathematics*, **54**:450–469 (2005). Having described the details of the generalized algorithm it is validated on two test cases from the original paper and is then applied to one-phase and, for the first time, two-phase Stefan problems in one and two space dimensions, paying particular attention to the implementation of the interface boundary conditions. Results are presented to demonstrate the accuracy and the effectiveness of the method, including comparisons against analytical solutions where available.

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1 Introduction

Moving-mesh methods have been used for the solution of partial differential equations (PDEs) in recent years in various ways. The motivation is usually to improve the resolution of solutions [4, 14, 17, 19, 20, 29, 38], to track special features of a solution (such as shocks, singularities and moving boundaries) [2, 5–7, 10, 18, 30–33, 42–44], and/or to

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exploit geometric properties such as scale-invariance, orderings or asymptotics [8,12,15]. The moving-mesh methods can be divided into two broad categories: those based on mappings between a fixed computational mesh and physical space [10, 16, 17, 38] and those based on velocities expressed in terms of the mesh coordinates in physical space [6, 12, 18–20, 34]. In this paper we focus on one particular velocity-based moving-mesh finite element method [6–8], which is related to the Geometric Conservation Law [18,40]. This method has been successfully applied to a range of time-dependent nonlinear PDE problems involving singularities and implicit moving boundaries: however, since its original introduction in [6], a number of improvements have been made which are incorporated into the description below.

The algorithm has been designed to be very general in nature and applicable to a large family of problems. Having validated the proposed method for a range of situations, the main thrust of this paper is its application, for the first time, to phase-change problems involving an internal moving boundary. Many numerical schemes have been applied to such problems, including those involving moving-mesh techniques [10, 25, 26, 33, 39]. The application of the moving-mesh finite element method described in this paper to this problem is however new, and is shown to be both accurate and robust.

The layout of the paper is as follows. Section 2 gives a description of the moving-mesh finite element method of [6] incorporating a number of recent developments. Section 3.1 presents validating results from a mass-conserving problem using the porous medium equation [41], which is a second order nonlinear diffusion equation for which simple self-similar solutions exist with finite support that grows with time. Section 3.2 then describes validating results from a non-mass-conserving implicit moving boundary problem, a simple model of absorption and diffusion, referred to here as the Crank-Gupta problem [24], which contains a sink term which causes the finite support to shrink with time. Section 4 contains the main application of the moving-mesh finite element method to one-phase and two-phase Stefan problems and Section 5 contains details of the results of numerical experiments, validated against analytic solutions whenever possible. Finally, Section 6 includes a discussion of points raised in the paper.

2 A moving-mesh finite element method

In this section we present a derivation of the equations used by the moving-mesh finite element method to find mesh velocity potentials which are subsequently used to define the nodal velocities. This is followed by a summary of the complete algorithm, as used in this paper, and the enhancements which have been made since its original publication in [6].

2.1 A mesh movement velocity potential

Let $u(\mathbf{x},t)$ be the solution of a well-posed time-dependent nonlinear PDE problem of general form